ECE 4960

Prof. Kirstin Hagelskjær Petersen
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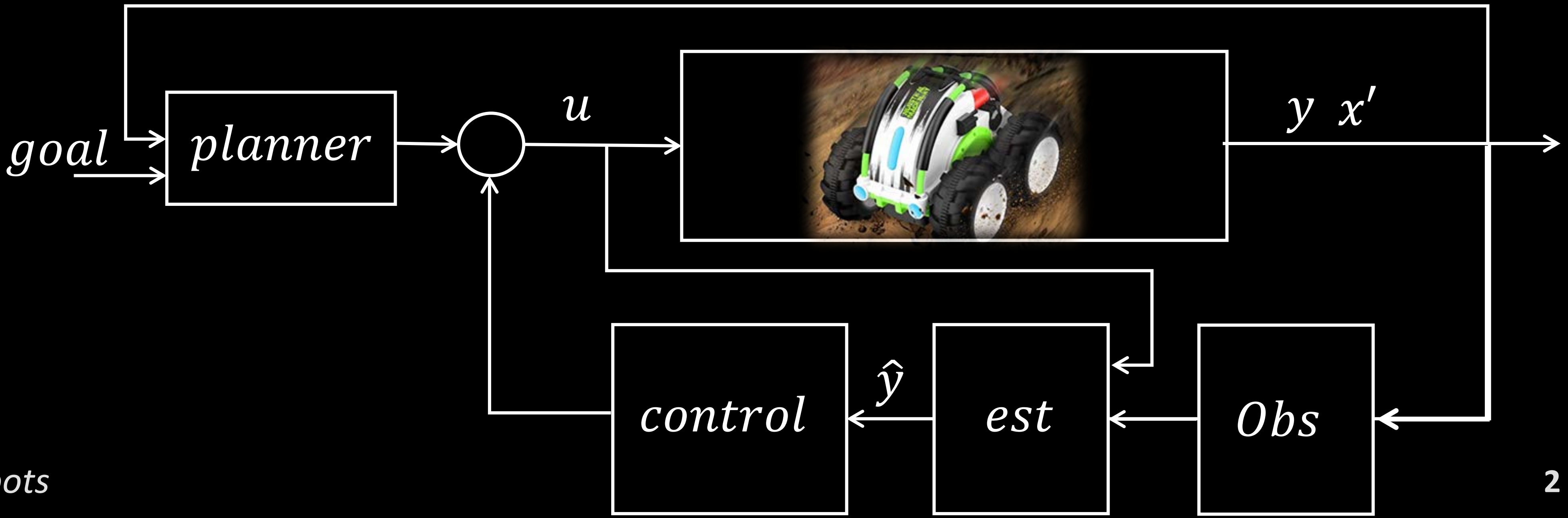
Fast Robots

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Inverted pendulum dynamics

$$\dot{x} = Ax + Bu$$



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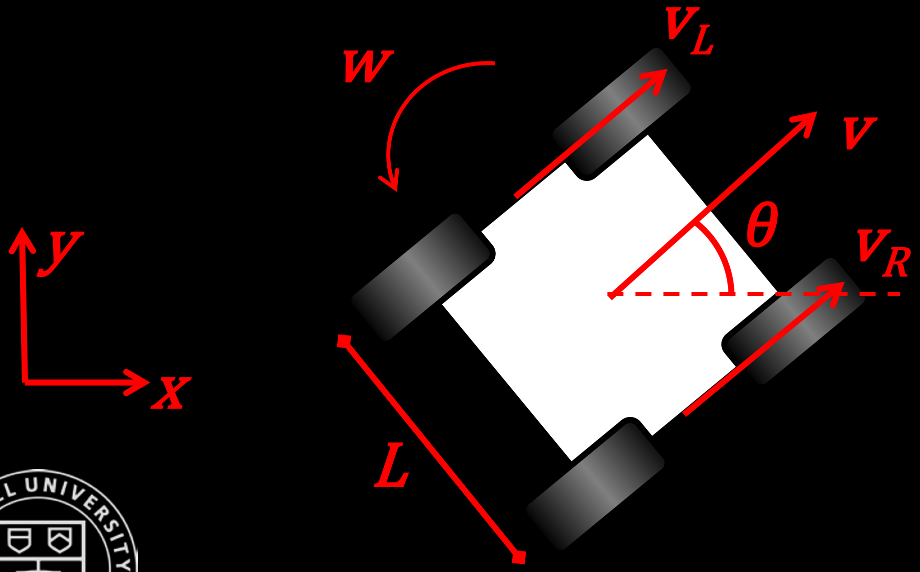

$$\dot{x} = cos(\theta)v$$
$$\dot{y} = sin(\theta)v$$
$$\dot{\theta} = w$$

$$\Rightarrow$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos(\theta)v & 0 \\ sin(\theta)v & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}}$$

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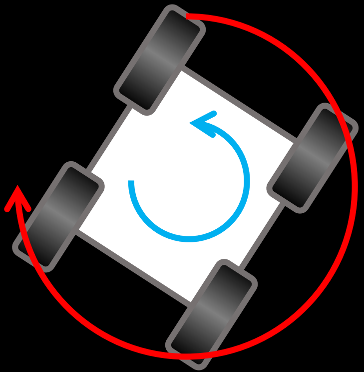
Lecture 6 - PID



1st order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$

2nd order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ cst & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

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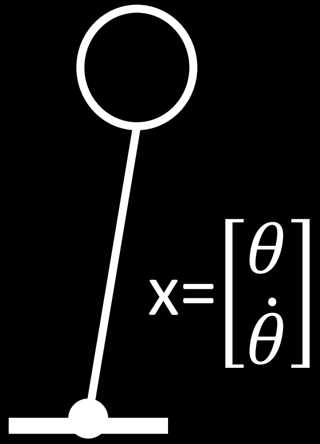


This should look familiar from..
- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

Linear Systems

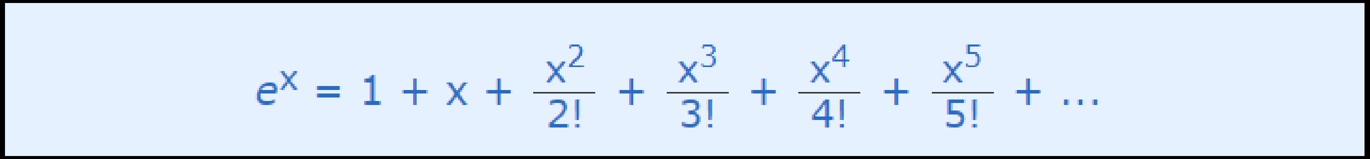
Linear system

$$\overset{\text{vector}}{\dot{x}} = \overset{\text{matrix}}{A}x$$

Basic solution

$$x(t) = \boxed{e^{At}}x(0)$$

Taylor series expansion



$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Intuition for eigenvectors (and stability)…

Linear system $\overset{\text{vector}}{\dot{x}} = \overset{\text{matrix}}{A}x$

Basic solution $x(t) = e^{At}x(0)$

Map the system to eigenvector coordinates to make computation easier

- Apply a linear transform: $z = Tx \quad \Leftrightarrow x = T^{-1}z$
- Substitute into the original equation: $T^{-1}\dot{z} = AT^{-1}z \quad \Leftrightarrow \dot{z} = TAT^{-1}z$
- Pick the matrix, T, such that TAT^{-1} becomes simpler than A

Eigenvectors

Eigenvectors and Eigenvalues

matrix

vector

$$A\xi = \lambda\xi$$

scalar number (eigenvalue)

- Eigenvectors, ξ, of A
- Matrix of eigenvectors, T

$$T = [\xi_1 \quad \xi_2 \quad \cdots \quad \xi_n]$$

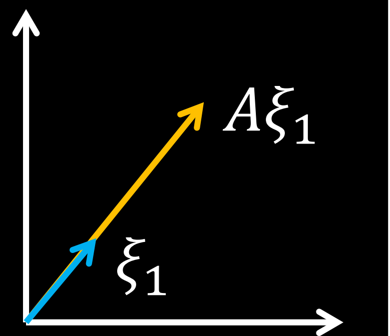
- Diagonal matrix of eigenvalues, D

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Matlab $\gg [T, D] = eig(A);$

$$\boxed{AT = TD}$$

- $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$
- $\xi_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\lambda_1 = 4$



z-coordinates versus x-coordinates

$$\dot{x} = Ax \qquad x = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$AD = TD \quad \Leftrightarrow T^{-1}AT = D$$

in the eigenvector directions

$$x = Tz$$

$$\dot{x} = T\dot{z} \quad = Ax$$

$$T\dot{z} = ATz$$

$$\dot{z} = T^{-1}ATz$$

$$\boxed{\dot{z} = Dz}$$

→ By mapping our system to eigenvector coordinates, the dynamics become diagonal (very simple!)

z-coordinates versus x-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$T^{-1}AT = D$$

$$\dot{z} = Dz$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$z_1(t) = e^{\lambda_1 t} z_1(0) \quad \dots \quad z_n(t) = e^{\lambda_n t} z_n(0)$$

$$z(t) = e^{Dt} z(0) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} z(0)$$

…it is much simpler to think about your system in eigenvector coordinates!

z-coordinates versus x-coordinates

$$\boxed{\begin{aligned}\dot{x} &= Ax = T\dot{z}\\ T^{-1}AT &= D\\ \dot{z} &= Dz\end{aligned}}$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = TT^{-1} + TDT^{-1}t + \underbrace{(TD\underbrace{T^{-1}T}_{I}DT^{-1})}_{TD^2T^{-1}}\frac{t^2}{2!} + \ldots$$

z-coordinates versus x-coordinates

$$\boxed{\begin{gathered}\dot{x} = Ax = T\dot{z}\\ T^{-1}AT = D\\ \dot{z} = Dz\end{gathered}}$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = TT^{-1} + TDT^{-1}t + (TDT^{-1}TDT^{-1})\frac{t^2}{2!} + \ldots$$

$$e^{At} = T\left[I + Dt + \frac{D^2t^2}{2!} + \cdots + \frac{D^nt^n}{n!}\right]T^{-1} = T\underset{\uparrow}{e^{Dt}}T^{-1}$$

easy to compute!

z-coordinates versus x-coordinates

$$\dot{x} = Ax$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

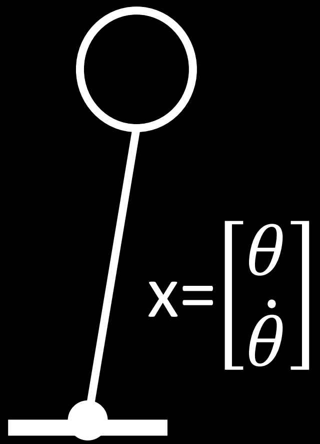
$$AD = TD$$

$$T^{-1}AT = D$$

$$x = Tz$$

system solution in physical coordinates

$$x(t) = \underbrace{T\underbrace{e^{Dt}\underbrace{T^{-1}x(0)}_{z(0)}}_{z(t)}}_{x(t)}$$



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Eigenvalues and Stability

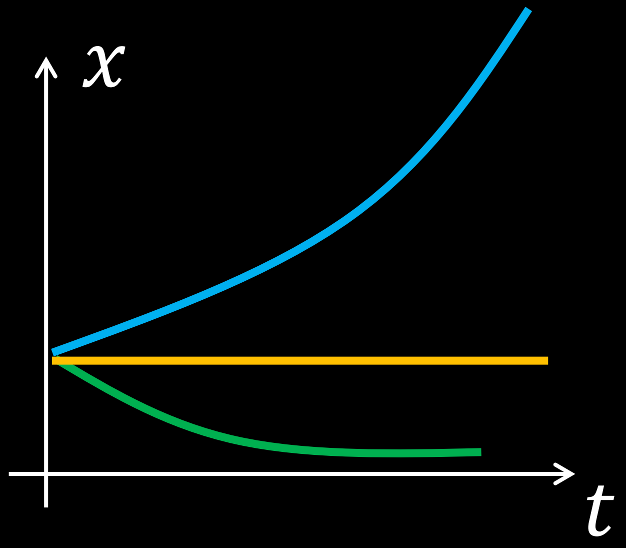
Stability (Continuous Time)

$$\dot{x} = Ax$$

$$x(t) = T\boxed{e^{Dt}}T^{-1}x(0)$$

$$\gg [T, D] = la.eig(A);$$

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$



- If even one of the $e^{\lambda_n t}$ goes to ∞ all go to ∞
- Complex eigenvalues
  - $\lambda = a \pm ib$
- Euler's formula
  - $e^{\pm\lambda t} = e^{at}[\overbrace{\cos(bt) \pm i\sin(bt)}^{=1}]$

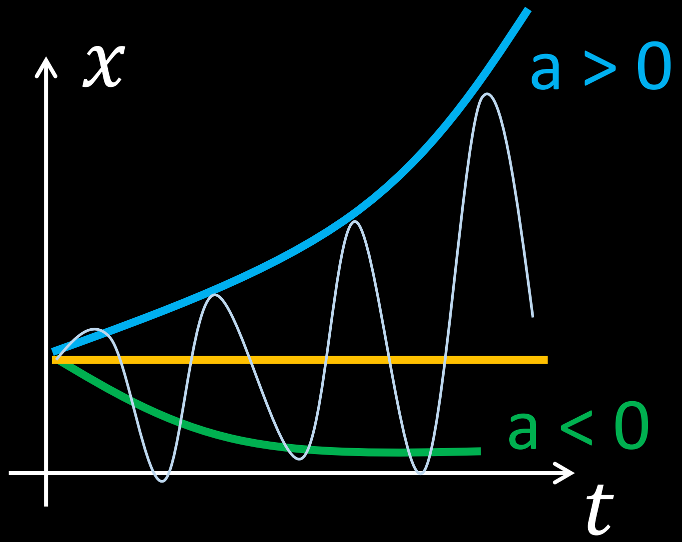
Stability (Continuous Time)

$$\dot{x} = Ax$$

$$x(t) = T\boxed{e^{Dt}}T^{-1}x(0)$$

$$\gg [T, D] = la.eig(A);$$

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$



- If even one of the $e^{\lambda_n t}$ goes to ∞ all go to ∞
- Complex eigenvalues
  - $\lambda = a \pm ib$
- Euler's formula
  - $e^{\pm\lambda t} = e^{at}[\overbrace{\cos(bt) \pm i\sin(bt)}^{=1}]$

- The system is stable iff all the real parts of all the eigenvalues are negative!

Stability (Continuous Time)

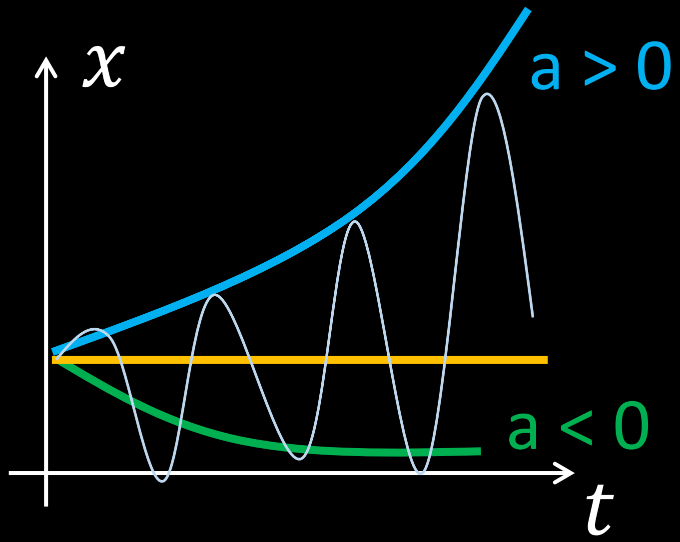
$$\dot{x} = Ax$$

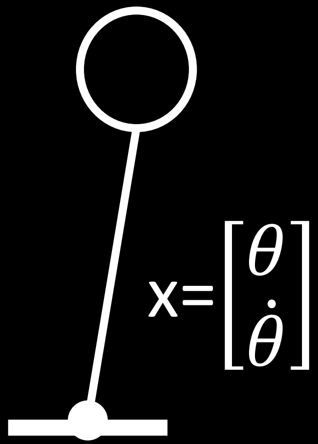
$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$\gg [T, D] = la.eig(A);$$

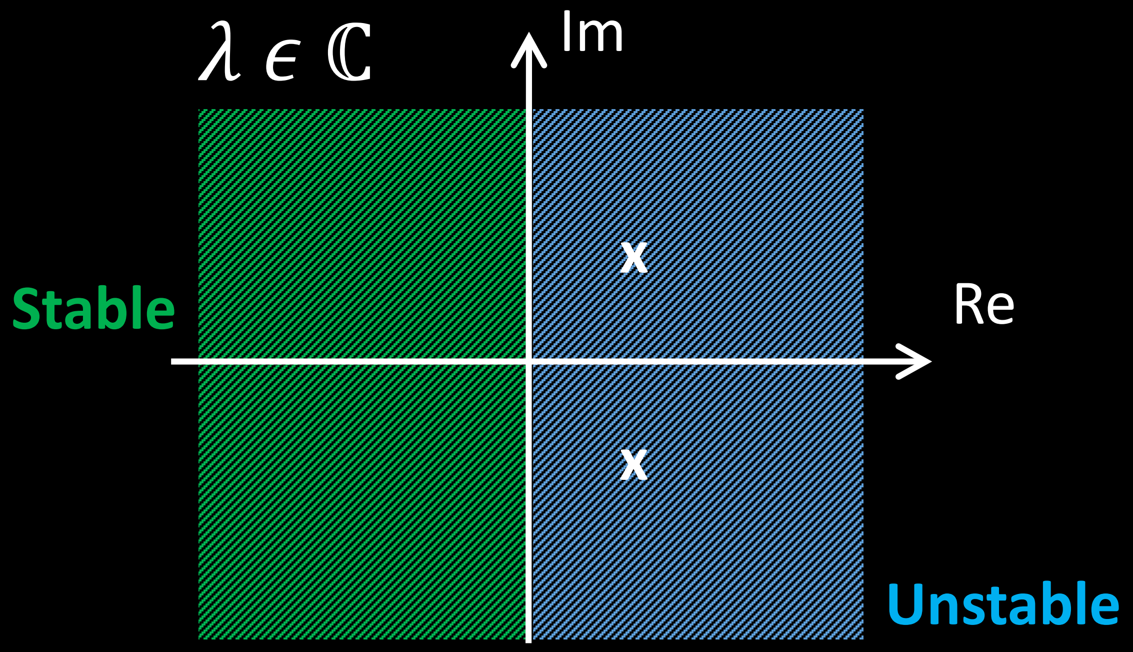
$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

$$\lambda = a + ib$$

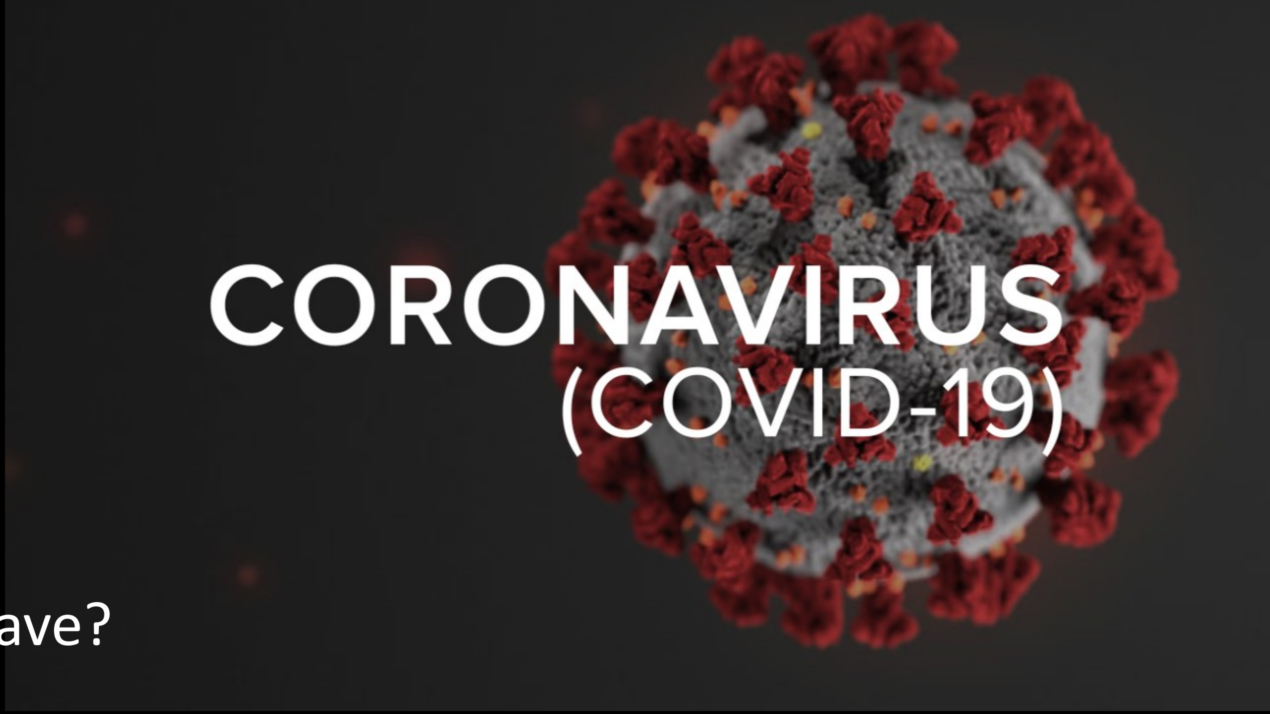




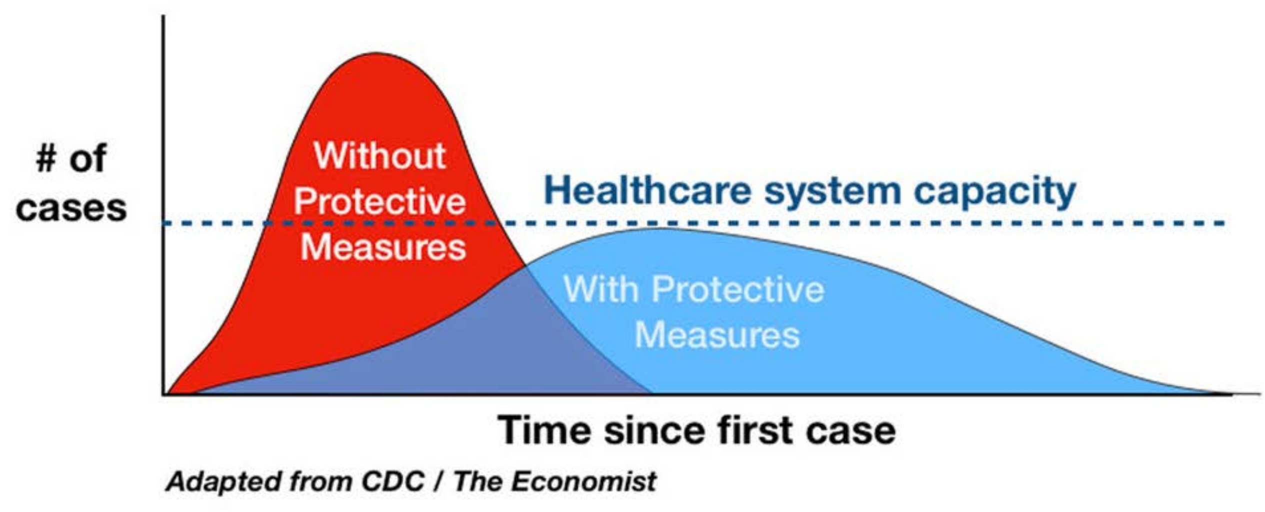
$\lambda \in \mathbb{C}$



Stability (Continuous Time)



- COVID-19, very simplified example
- What is our state vector, x?
  - $x = [\#infected\ people]$
- System
  - $\dot{x} = Ax$
- How many eigenvectors does the system have?
  - 1
- Is the system unstable?
  - The eigenvalue of A / A has a positive real part
- What are our control inputs?
  - Wearing masks
  - Social distancing
  - Election…



Adapted from CDC / The Economist

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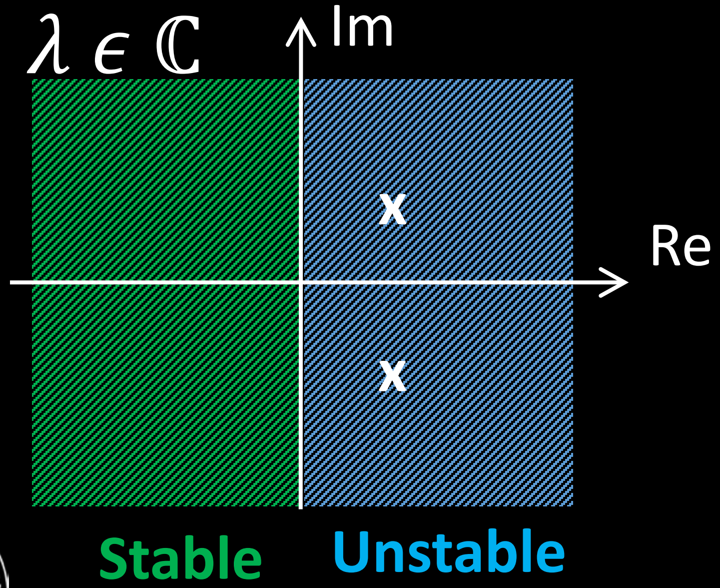
Discrete Time Systems

Stability (Discrete Time)

$$\dot{x} = Ax$$

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda \epsilon \mathbb{C}$



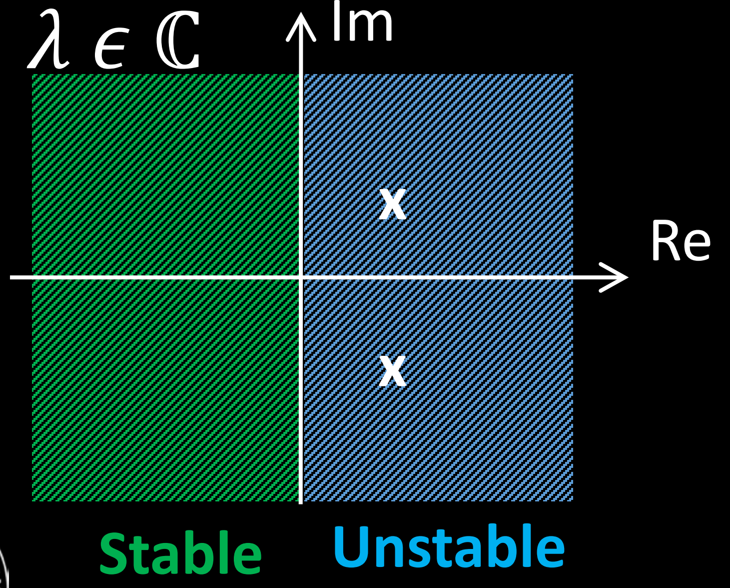
- $x(k+1) = \tilde{A}x(k) \quad , x(k) = x(k\Delta t)$
- $\tilde{A} = e^{A\Delta t}$
- $x_1 = \tilde{A}x_0 \qquad \tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1} \qquad \tilde{\lambda}$
- $x_2 = \tilde{A}x_1 = \tilde{A}^2x_0 \qquad \tilde{A}^2 = \tilde{T}\tilde{D}^2\tilde{T}^{-1} \qquad \tilde{\lambda}^2$
- $x_3 = \tilde{A}^3x_0 \qquad \tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1} \qquad \tilde{\lambda}^3$
- … … …
- $x_n = \tilde{A}^nx_0 \qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} \qquad \tilde{\lambda}^n$

Stability (Discrete Time)

$$\dot{x} = Ax$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$



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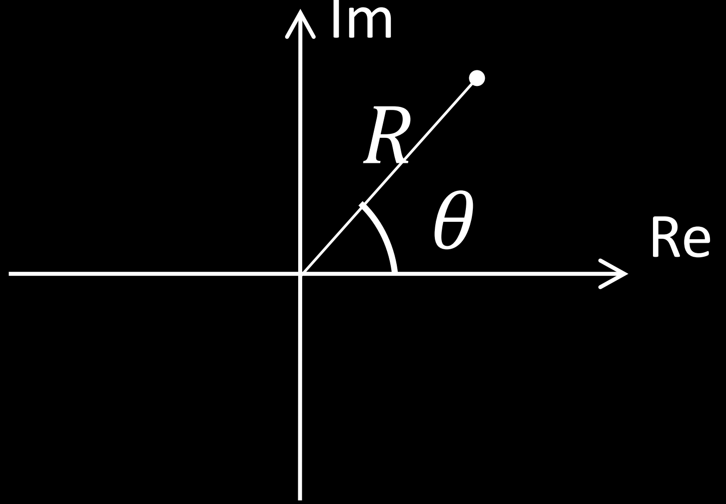
- $x_3 = \tilde{A}^3x_0 \qquad \tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1} \qquad \tilde{\lambda}^3$
- $\dots \qquad \dots \qquad \dots$
- $x_n = \tilde{A}^nx_0 \qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} \qquad \tilde{\lambda}^n$

- $\tilde{\lambda} = Re^{i\theta}$
- $\tilde{\lambda}^n = R^ne^{in\theta}$



Stability (Discrete Time)

$$\dot{x} = Ax$$

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n \end{bmatrix}$$

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- $\ldots \qquad \ldots \qquad \ldots$
- $x_n = \tilde{A}^n x_0 \qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} \qquad \tilde{\lambda}^n$

- $\tilde{\lambda} = Re^{i\theta}$
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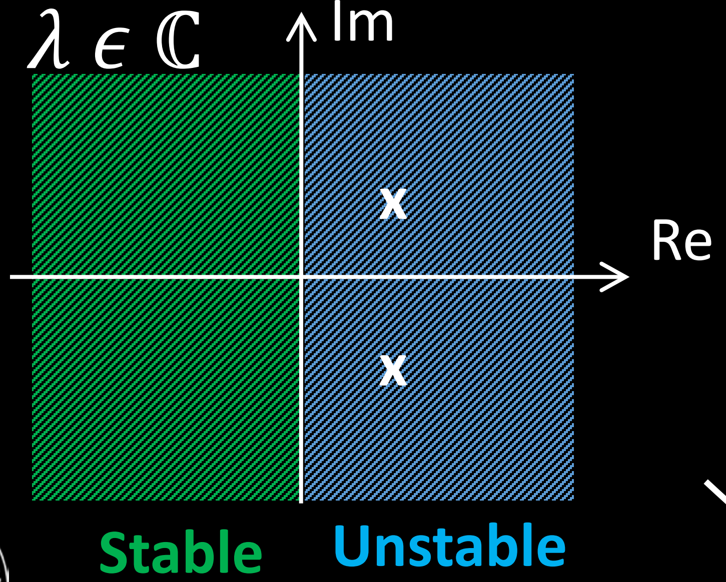

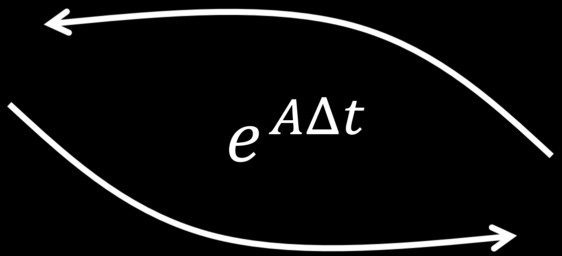

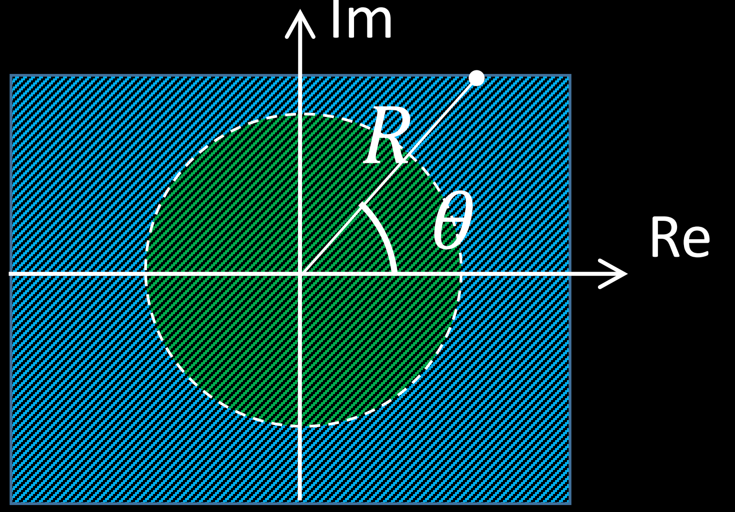

Stability (Discrete Time)

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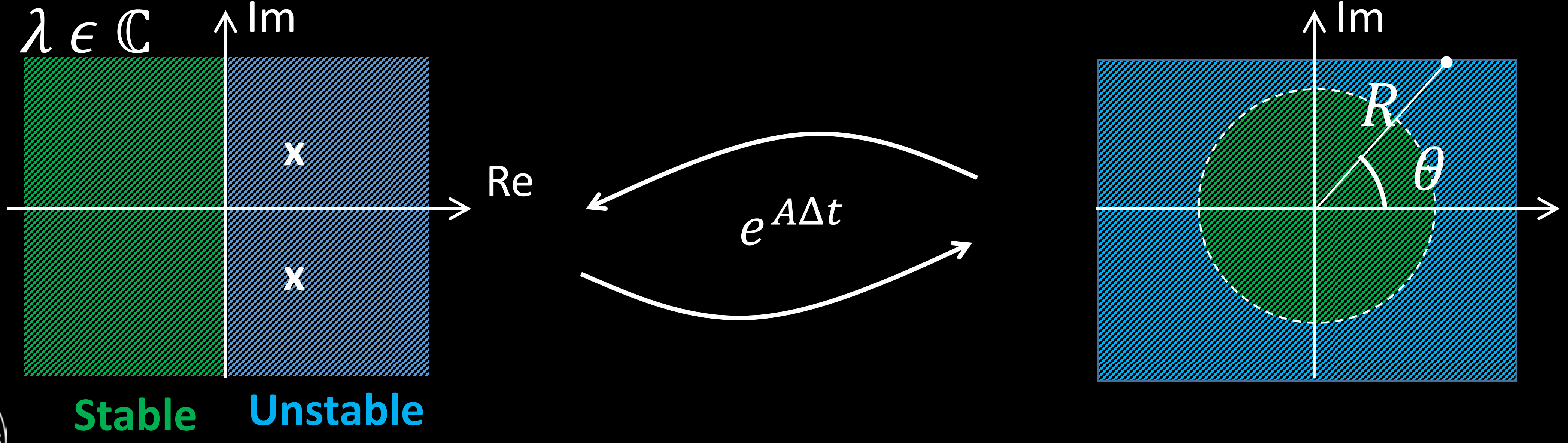
$$\tilde{A} = e^{A\Delta t}$$

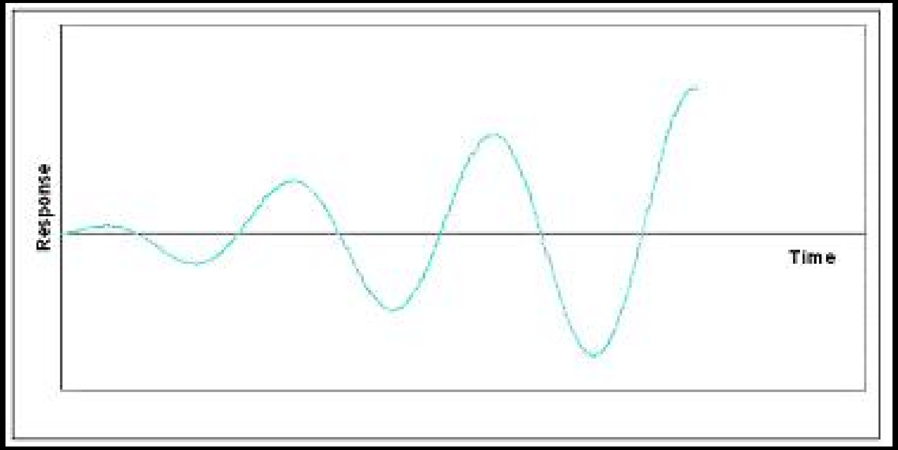
$$\tilde{\lambda}^n = R^n e^{in\theta}$$

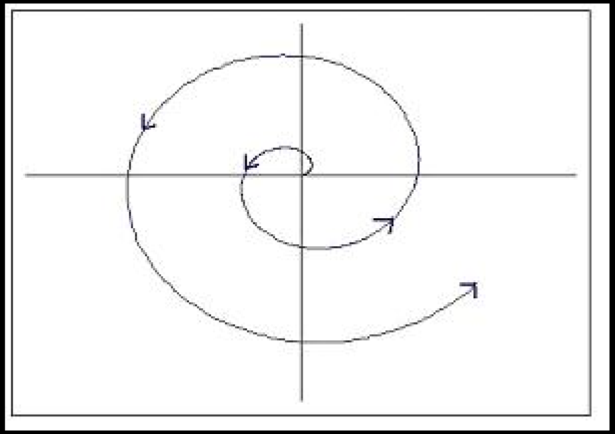


Stability (Discrete Time)

$\dot{x} = Ax$

$x(k+1) = \tilde{A}x(k)$



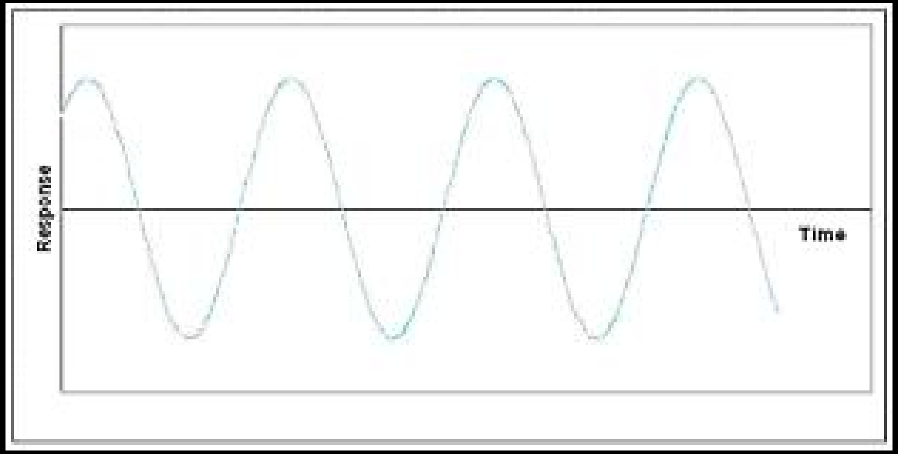


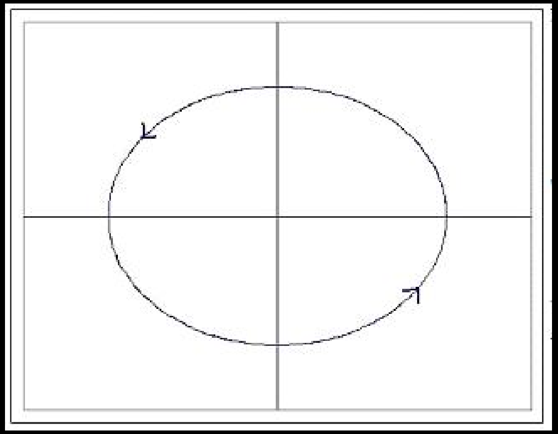
Unstable
(Positive real part)

Stability (Discrete Time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k)$$





Critically stable
(Zero real part)

Stability (Discrete Time)

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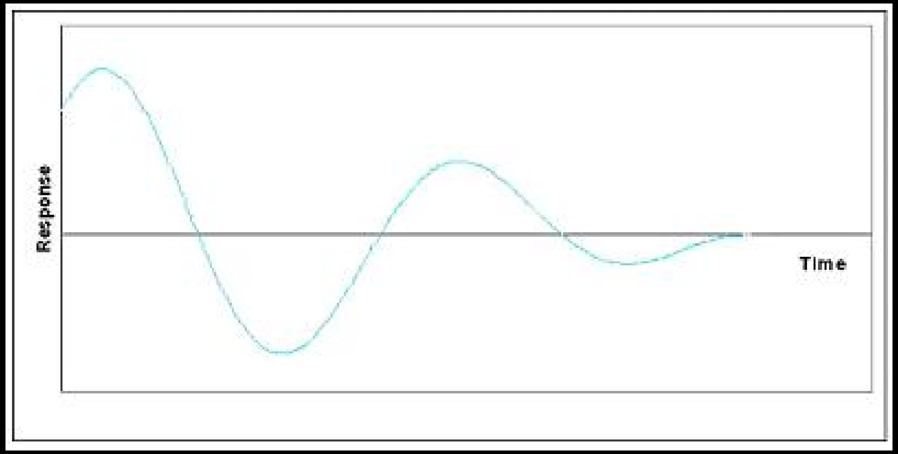
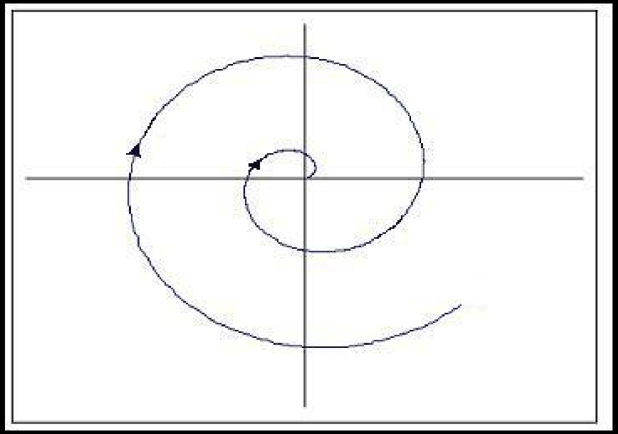
$$x(k+1) = \tilde{A}x(k)$$





Stable
(Negative real part)

Linear Systems

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

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$$\dot{x} = Ax + Bu$$

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- ECE3250 Signals and systems
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- MAE3260 System Dynamics
- etc…