Prof. Kirstin Hagelskjær Petersen
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Fast Robots

Linear Systems

Based on "Control Bootcamp", Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Inverted pendulum dynamics

$$\dot{x} = Ax + Bu$$

This should look familiar from..

- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

Linear Systems – “review of review”

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$
- Linear transform: $AT = TD$
- Solution: $e^{At} = Te^{Dt}T^{-1}$
- Mapping from z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff a<0
  - Discrete time: $x(k+1) = \tilde{A}x(k), \tilde{A} = e^{A\Delta t}$
    - Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff R<1

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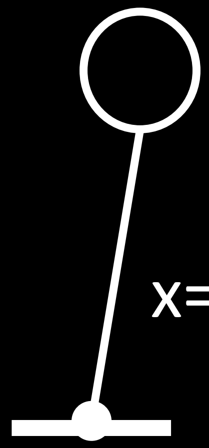

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

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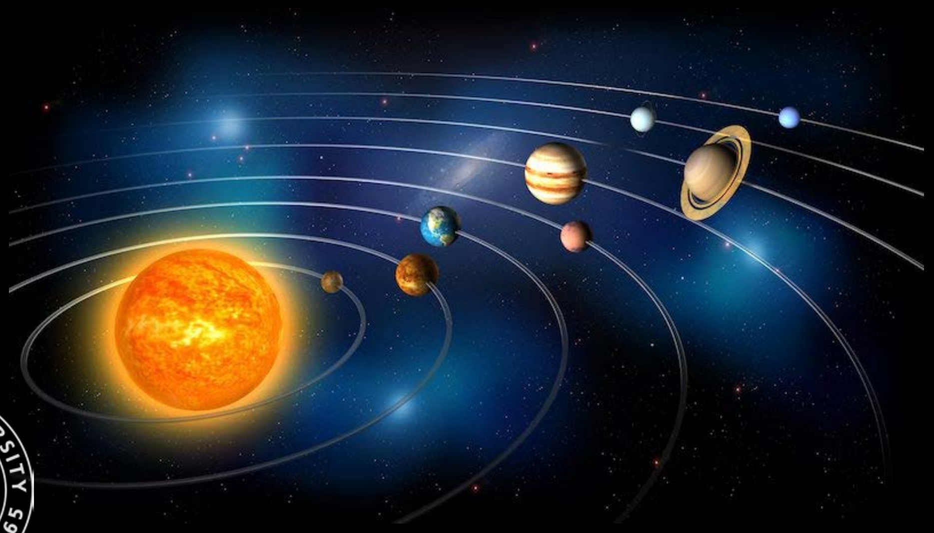
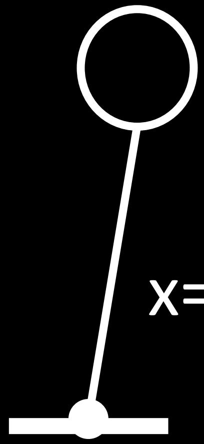
Linearizing Non-Linear Systems

Basic Steps to linearize a nonlinear system

1. Find some fixed points
   - $\bar{x}\ \ s.t.\ f(\bar{x}) = 0$
   - (basically points where the system doesn't move)
2. Linearize about $\bar{x}$
   - $\frac{Df}{Dx}|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$ ← *"Jacobian"*





$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{x} = f(x) \quad \Rightarrow \quad \dot{x} = Ax$$

Example

$$\dot{x_1} = f_1(x_1, x_2) = x_1 x_2$$

$$\dot{x_2} = f_2(x_1, x_2) = {x_1}^2 + {x_2}^2$$

$$\frac{Df}{Dx} = \begin{bmatrix} {\partial f_1}/{\partial x_1} & {\partial f_1}/{\partial x_2} \\ {\partial f_2}/{\partial x_1} & {\partial f_2}/{\partial 2} \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$

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- If you zoom in on $\bar{x}$, your system will look linear!

$$\dot{x} = f(x) \quad \Rightarrow \quad \dot{x} = Ax$$

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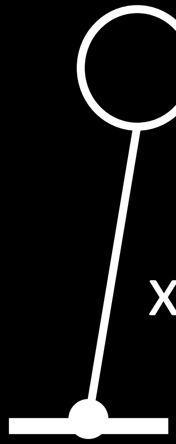

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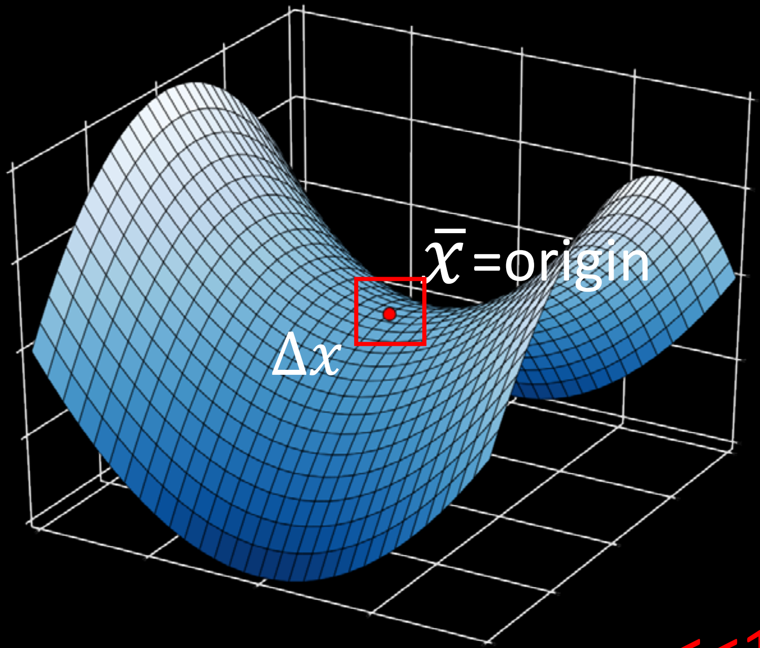

$$\dot{x} = f(x)$$

$$\dot{x} = \underbrace{f(\bar{x})}_{0} + \frac{Df}{Dx}|_{\bar{x}}(x-\bar{x}) + \underbrace{\frac{D^2 f}{D^2 x}|_{\bar{x}}(x-\bar{x})^2 + \frac{D^3 f}{D^3 x}|_{\bar{x}}(x-\bar{x})^3 + \cdots}_{<<1}$$

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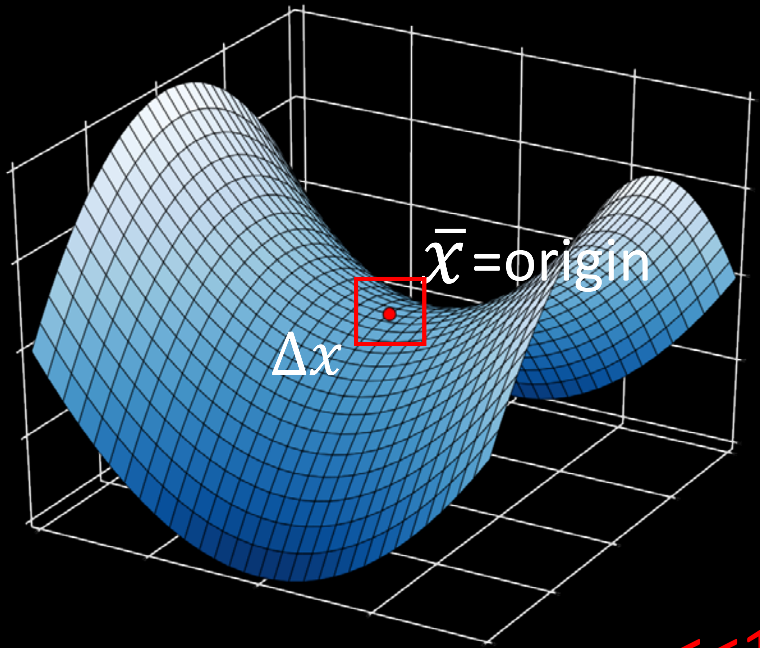

- Good control will keep you close to the fixed point, where your model is valid!

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$$\Delta\dot{x} = \frac{Df}{Dx}|_{\bar{x}}\Delta x$$

$$\Rightarrow \Delta\dot{x} = A\Delta x$$

Linearizing Non-Linear Systems

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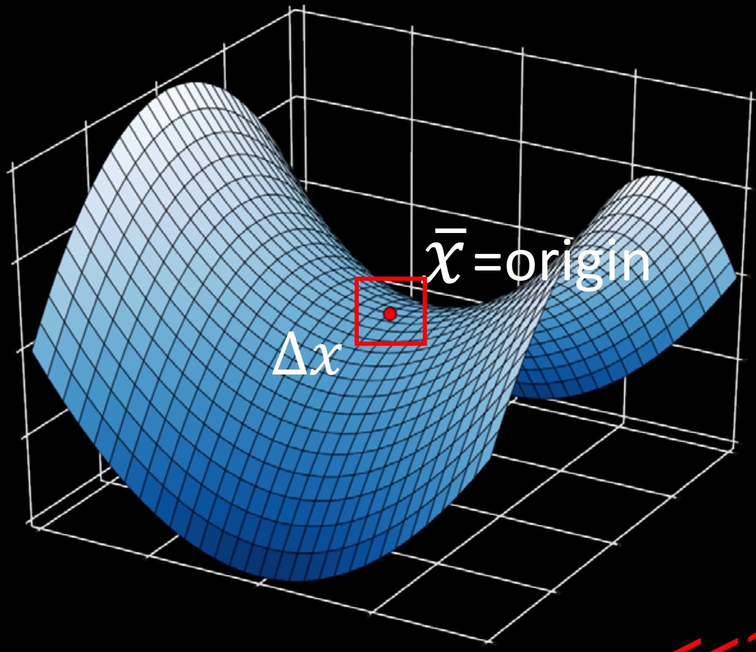

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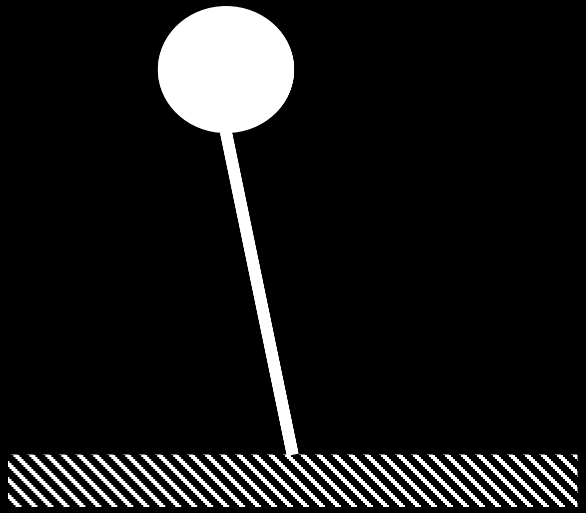

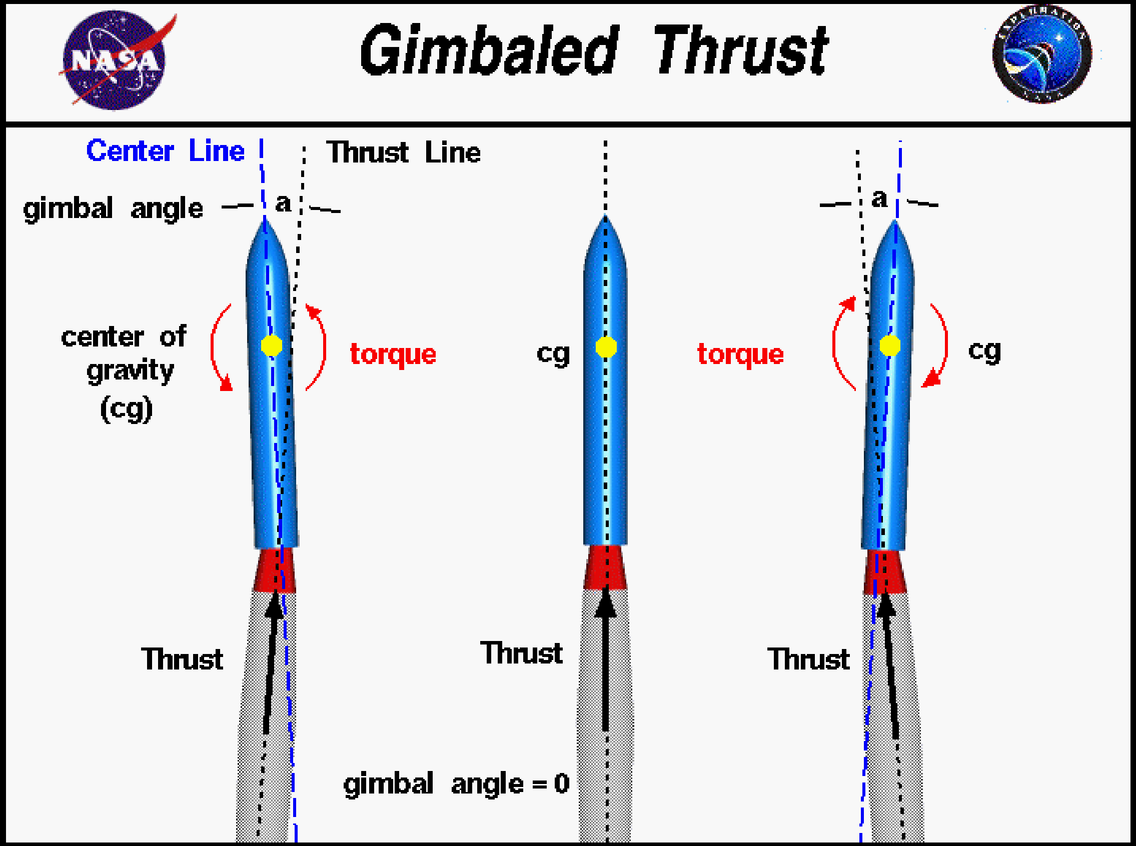

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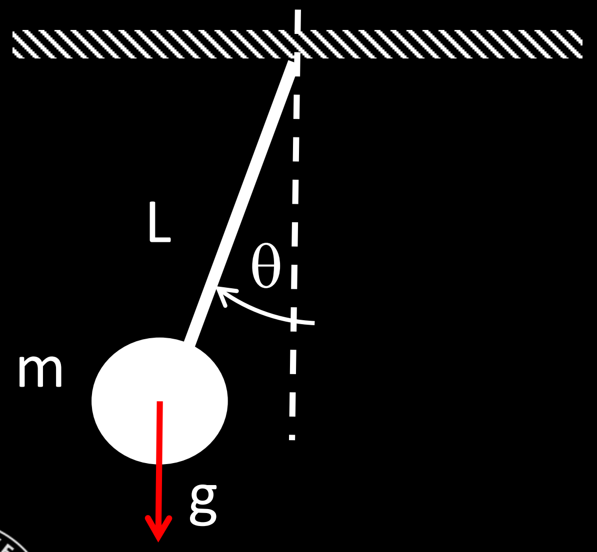

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Eq. of motion

- $\tau = -mgLsin(\theta)$
- $\tau = I\ddot{\theta}$
- $I\ddot{\theta} = -mgLsin(\theta)$
- Point mass inertia
  - $I = mL^2$
- $mL^2\ddot{\theta} = -mgLsin(\theta)$
- $\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \delta\dot{\theta}$

friction

Linearizing Non-Linear Systems

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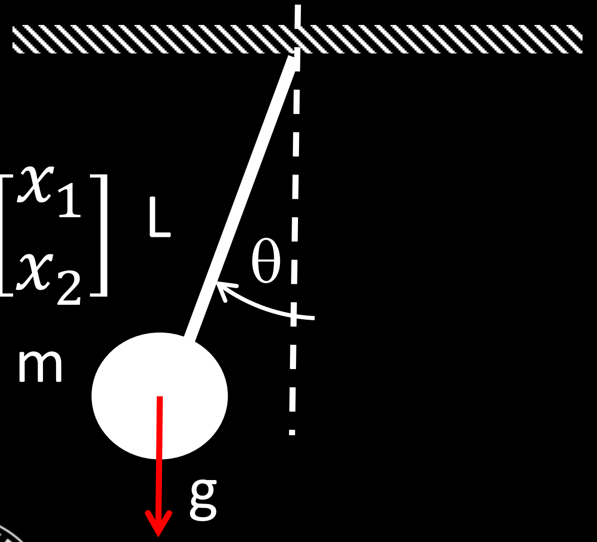
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



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$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \delta\dot{\theta}, \qquad \frac{g}{L} = 1$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \\ 0 \end{bmatrix}$$

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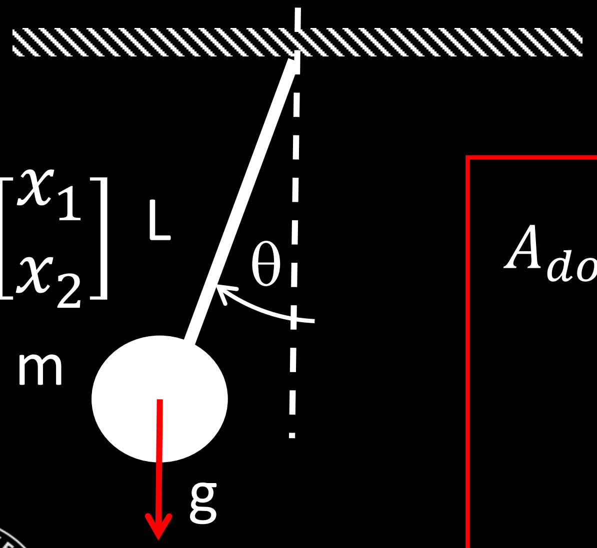

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0, \pi \\ 0 \end{bmatrix}$$

$$\frac{DF}{Dx} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$$

$$\frac{DF}{Dx} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -\delta \end{bmatrix}$$

$$A_{down} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$$

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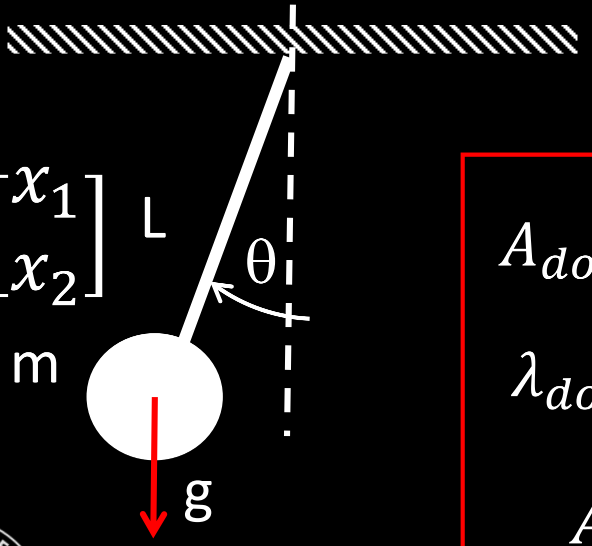

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$\lambda_{down} = \pm i$ stable!

$$A_{up} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}$$

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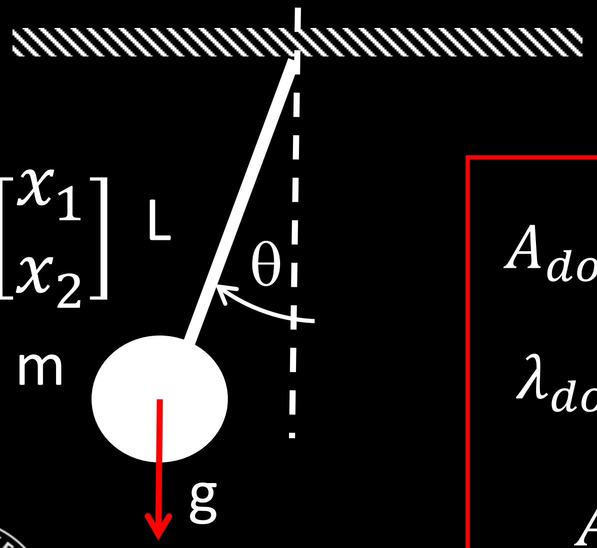

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Controllability

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

$$\underline{y} = \underline{\underline{C}}\,\underline{x}$$

$x \epsilon \mathbb{R}^{n}$

$A \epsilon \mathbb{R}^{n \times m}$

$u \epsilon \mathbb{R}^{q}$

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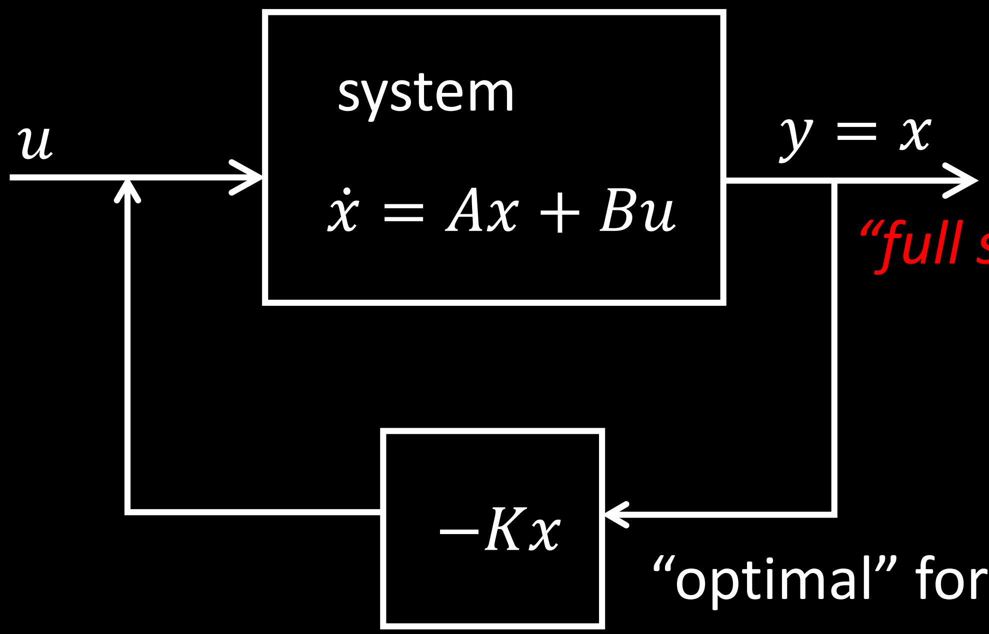


“full state feedback”

“optimal” for linear systems

Controllability

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = (A - BK)x$$

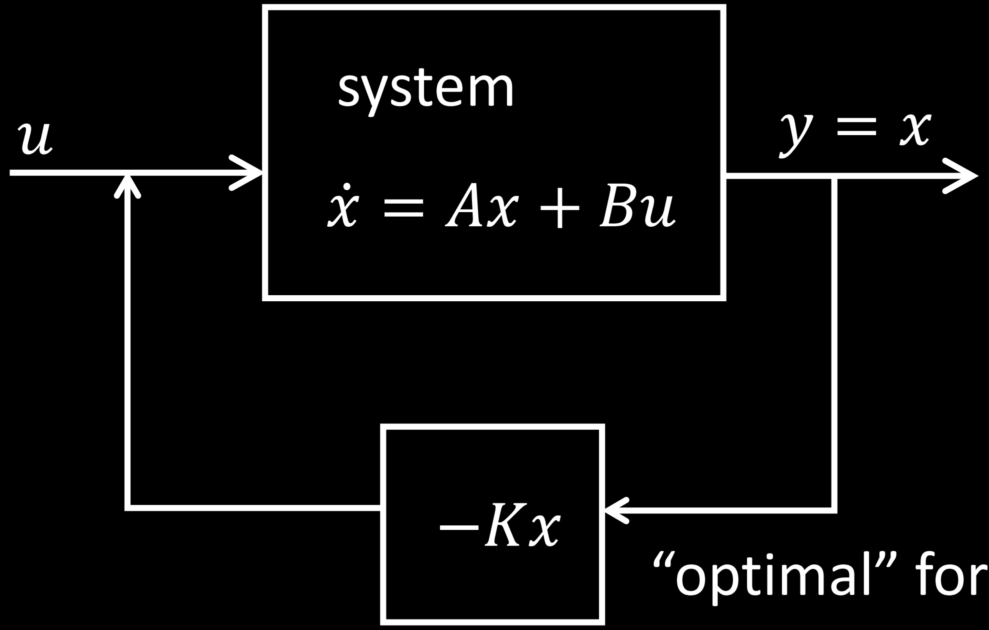
New dynamics

$x \in \mathbb{R}^n$

$A \in \mathbb{R}^{n \times m}$

$u \in \mathbb{R}^q$

$B \in \mathbb{R}^{n \times q}$



“optimal” for linear systems

A linear controller (K matrix) can be optimal for linear systems!

Controllability

- What determines whether or not a system is controllable?
  - A system is controllable, if you can steer your state x anywhere you want in $\mathbb{R}^n$

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

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$$\dot{x} = Ax - BKx$$

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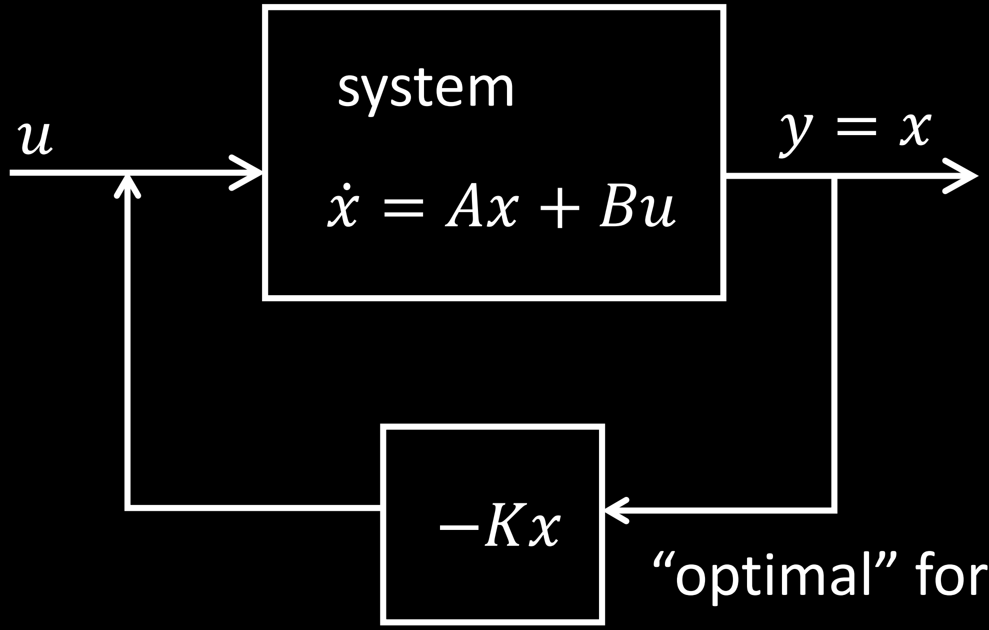

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$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

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  - Matlab/python >> ctrb(A,B)

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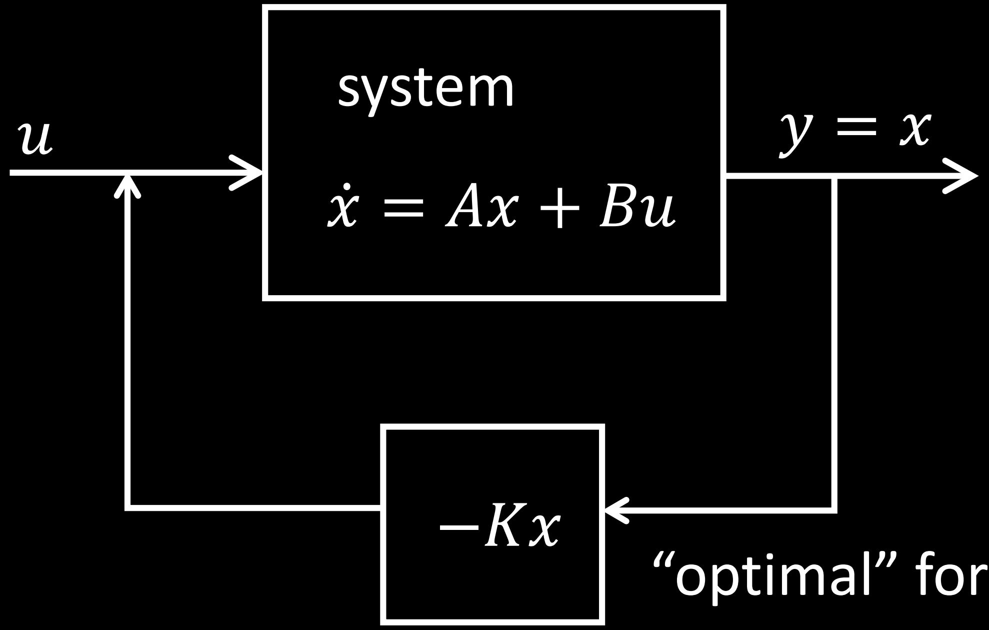

"optimal" for linear systems

Controllability

- Can you control this system?
  1. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *uncontrollable*
     - There's no way to directly/indirectly affect x_1
- What could you change to make it controllable?
  - Add more control authority!
  2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ *controllable*

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- Can you control this system?
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     - Systems with coupled can be controllable…
     - If A is tightly coupled, you can get away with a simple B

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- Matlab >> ctrb(A,B)
- Controllability matrix
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable

$$\underline{\dot{x}} = \underline{\underline{A}x} + \underline{\underline{B}u}$$

$$\dot{x} = (A - BK)x$$

$x \epsilon \mathbb{R}^n$

$A \epsilon \mathbb{R}^{n\times m}$

$u \epsilon \mathbb{R}^q$

$B \epsilon \mathbb{R}^{n\times q}$

Controllability

- Can you control this system?
  1. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *uncontrollable*
  2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ *controllable*
  3. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *controllable*
- Matlab >> ctrb(A,B)
- Controllability matrix
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable
- System 1:
  - $\mathbb{C} = \Big[\qquad \Big]$
  - $\mathbb{C} = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$ *rank=1, n=2*

← *These still can't touch x_1!*

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$

$$\dot{x} = (A - BK)x$$

$x \epsilon \mathbb{R}^n$

$A \epsilon \mathbb{R}^{n\times m}$

$u \epsilon \mathbb{R}^q$

$B \epsilon \mathbb{R}^{n\times q}$

Controllability

$$\underline{\dot{x}} = \underline{\underline{A}x} + \underline{\underline{B}u}$$

$$\dot{x} = (A - BK)x$$

$x \epsilon \mathbb{R}^n$

$A \epsilon \mathbb{R}^{n \times m}$

$u \epsilon \mathbb{R}^q$

$B \epsilon \mathbb{R}^{n \times q}$

- Can you control this system?
  1. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ *uncontrollable*
  2. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *controllable*
  3. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ *controllable*
- Matlab >> ctrb(A,B)
- Controllability matrix
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable
- System 1: $\mathbb{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ *rank=1, n=2*
- System 3:
  - $\mathbb{C} = \begin{bmatrix} 0 & 1 \cdot 0 + 1 \cdot 1 \\ 1 & 0 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ *rank=2, n=2*

Fyi!

- Just because a linearized, nonlinear system is uncontrollable, it can still be nonlinearly controllable!
- $\mathbb{C}$ can also tell you how controllable a system is!

Controllability

- What determines whether or not a system is controllable?
  - A system is controllable, if you can steer your state x anywhere you want in $\mathbb{R}^n$
  - Matlab/python >> ctrb(A,B)

$$\underline{\dot{x}} = \underline{\underline{A}x} + \underline{\underline{B}u}$$

$$\underline{y} = \underline{\underline{C}x}$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

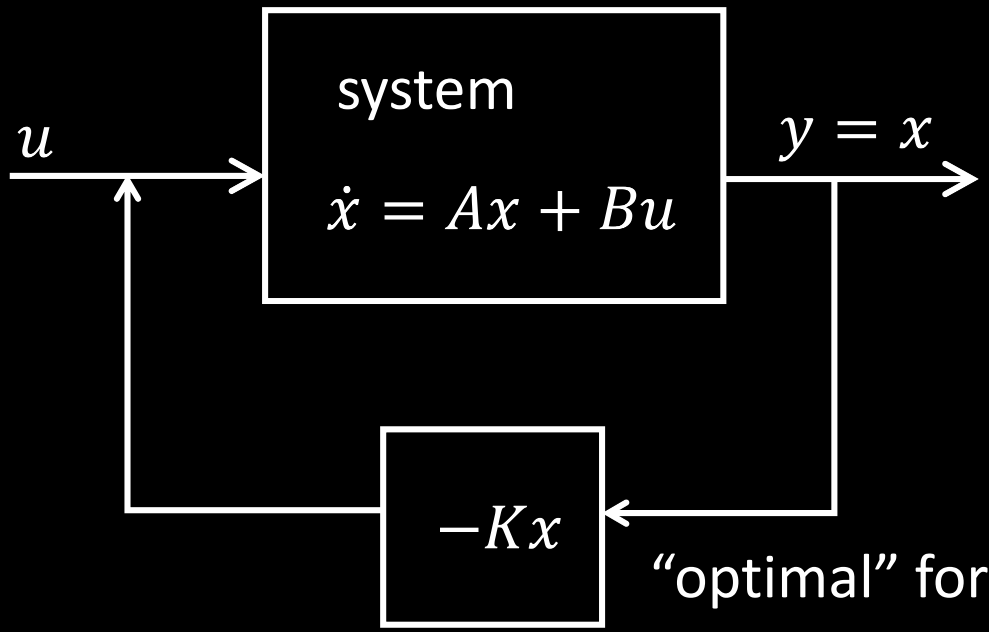
New dynamics

$x \epsilon \mathbb{R}^n$

$\mathrm{A} \epsilon \mathbb{R}^{n \times m}$

$\mathrm{u} \epsilon \mathbb{R}^q$

$\mathrm{B} \epsilon \mathbb{R}^{n \times q}$



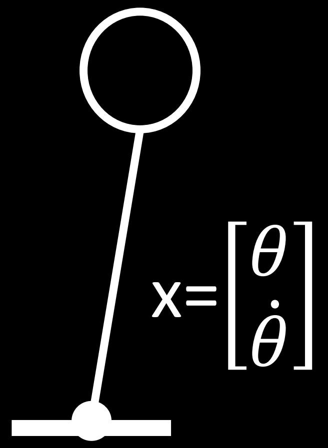
“optimal” for linear systems

Linear Systems

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Inverted pendulum dynamics

$$\dot{x} = Ax + Bu$$



This should look familiar from..
- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…