ECE 4960

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Fast Robots

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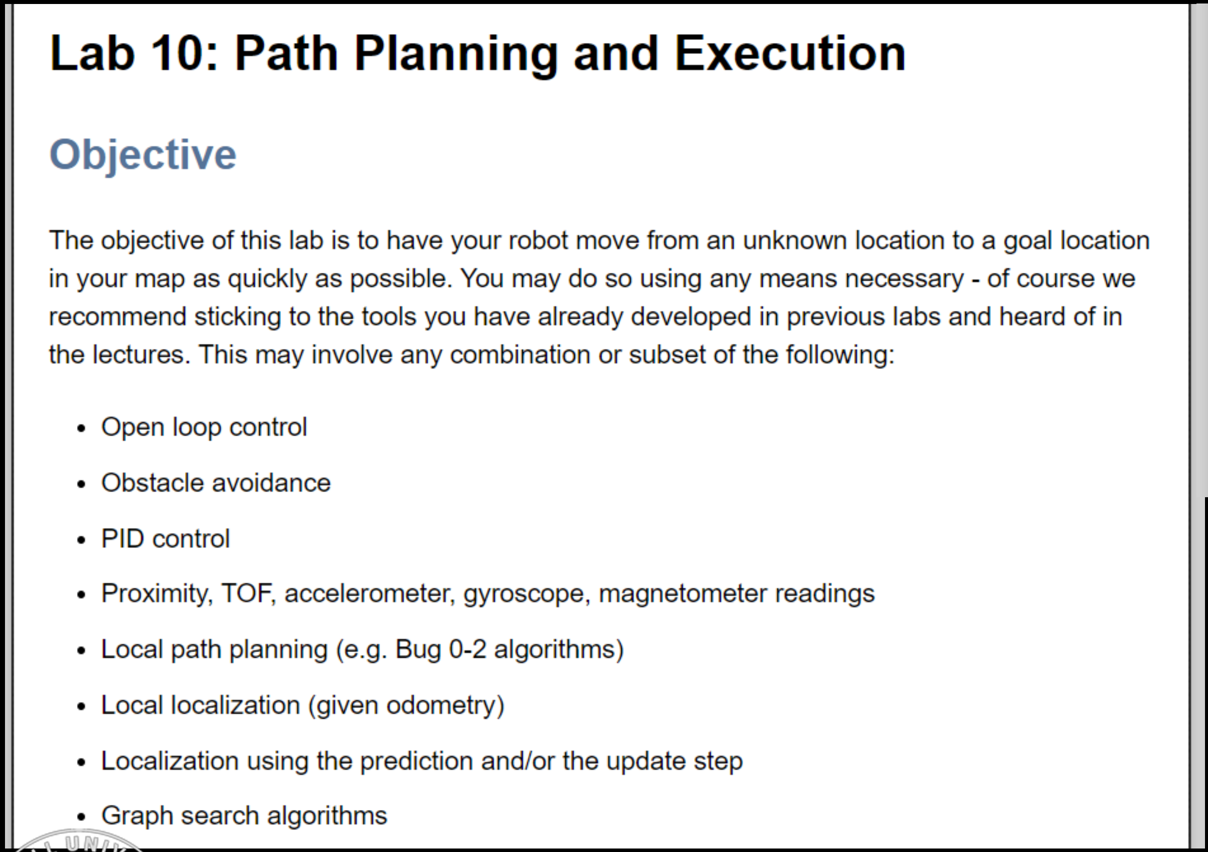
Questions on Lab 9/10?

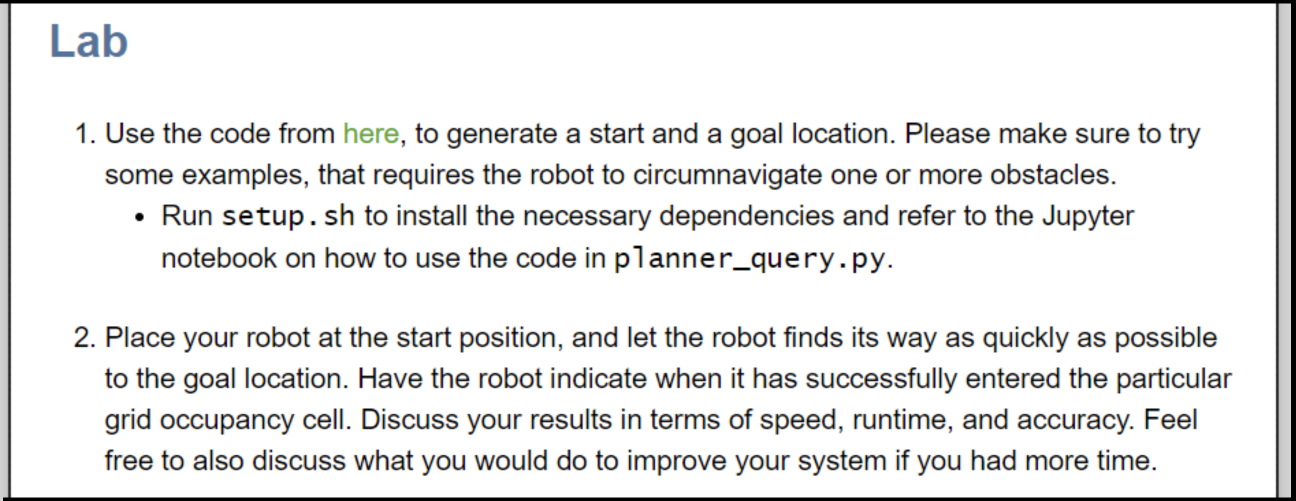


Lab 10: Path Planning and Execution

Objective

The objective of this lab is to have your robot move from an unknown location to a goal location in your map as quickly as possible. You may do so using any means necessary - of course we recommend sticking to the tools you have already developed in previous labs and heard of in the lectures. This may involve any combination or subset of the following:

- Open loop control
- Obstacle avoidance
- PID control
- Proximity, TOF, accelerometer, gyroscope, magnetometer readings
- Local path planning (e.g. Bug 0-2 algorithms)
- Local localization (given odometry)
- Localization using the prediction and/or the update step
- Graph search algorithms



Lab

1. Use the code from here, to generate a start and a goal location. Please make sure to try some examples, that requires the robot to circumnavigate one or more obstacles.
    - Run `setup.sh` to install the necessary dependencies and refer to the Jupyter notebook on how to use the code in `planner_query.py`.
2. Place your robot at the start position, and let the robot finds its way as quickly as possible to the goal location. Have the robot indicate when it has successfully entered the particular grid occupancy cell. Discuss your results in terms of speed, runtime, and accuracy. Feel free to also discuss what you would do to improve your system if you had more time.

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Review

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Inverted pendulum dynamics

$$\dot{x} = Ax + Bu$$

This should look familiar from..

- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

Review

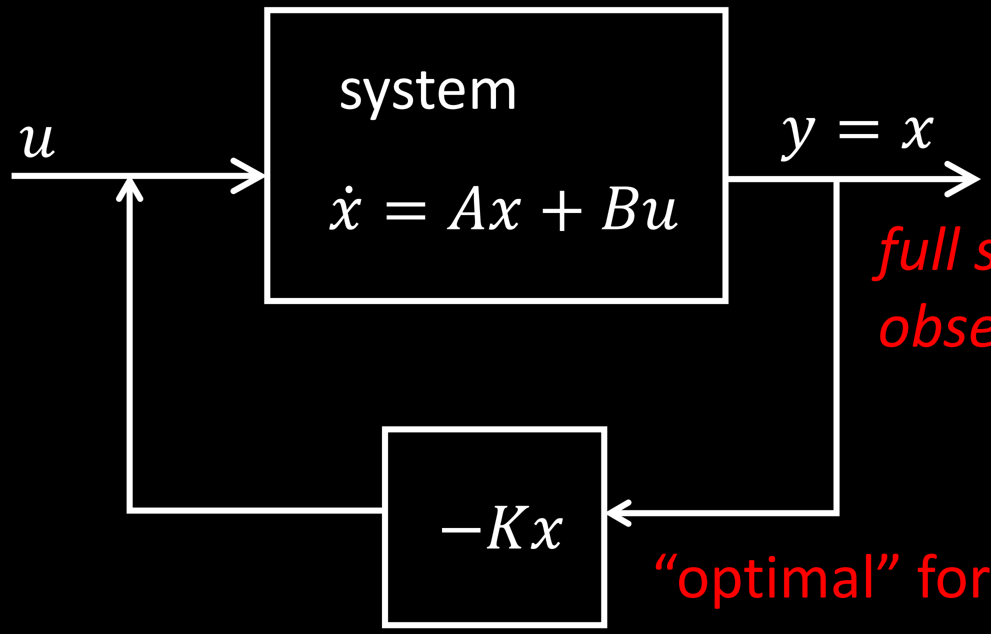
- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$

- Non-linear systems: $\dot{x} = f(x)$
- Linearization: $\frac{Df}{Dx}|_{\bar{x}}$
- Control: $\dot{x} = Ax + Bu$
- Controllability:
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$
  - $rank\big(ctrb(A,B)\big) = n$

- Linear transform: $AT = TD$
- Solution: $e^{At} = Te^{Dt}T^{-1}$
- Mapping from z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff a<0
  - Discrete time: $x(k+1) = \tilde{A}x(k), \tilde{A} = e^{A\Delta t}$
    - Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff R<1

```
>>[T,D] = eig(A)
>>rank(ctrb(A,B))
```

Review



full state observation

“optimal” for linear systems

$$\dot{x} = \underbrace{(A - BK)}_{\textit{New dynamics}}x$$

Review

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```
>>[T,D] = eig(A)
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Controllability Matrix and the Discrete Time Impulse Response

$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$

$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$

- Why does $\mathbb{C}$ predict controllability?!
- Discrete time impulse response: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$
  - $u(0) = 1 \qquad x(0) = 0$
  - $u(1) = 0 \qquad x(1) = \tilde{B}$
  - $u(2) = 0 \qquad x(2) = \tilde{A}\tilde{B}$
  - $u(3) = 0 \qquad x(3) = \tilde{A}^2\tilde{B}$
  - $\ldots \qquad \ldots$
  - $u(m) = 0 \qquad x(m) = \tilde{A}^{m-1}\tilde{B}$

If the system is controllable, then the impulse response affects every state in $\mathbb{R}^n$

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Reachability

Controllabillity and Reachability

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$
$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

Reachability

- $\mathcal{R}_t$, states that are reachable at time t
- $\mathcal{R}_t = \{\xi \epsilon \mathbb{R}^n$ for which there is an input $u(t)$ that makes $\mathrm{x(t)} = \xi$

Equivalences

1. The system is controllable
   - iff $rank(\mathbb{C}) = n$
2. You can choose K to arbitrarily place the eigenvalues of your closed loop system
   - $\dot{x} = (A - BK)x$
3. You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time
   - $\mathcal{R}_t = \mathbb{R}^n$

Controllabillity and Reachability

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$

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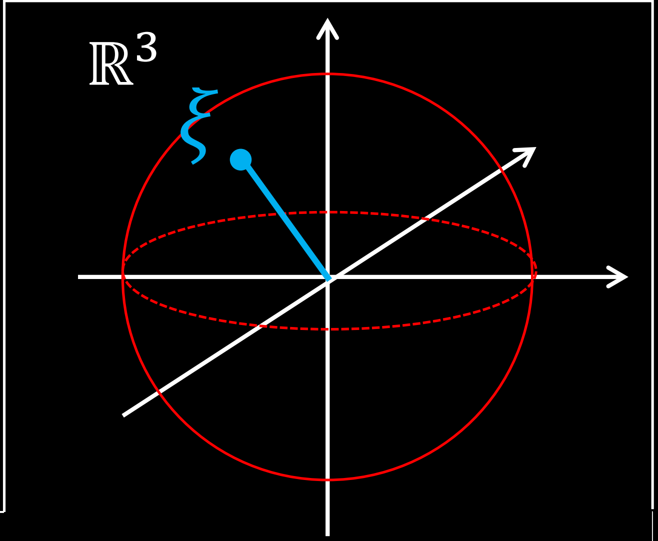

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   - $\mathcal{R}_t = \mathbb{R}^n$

*You cannot have more repeated eigenvalues than there are columns of B

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Controllability Gramians

Controllabillity Gramian

- $x(t) = e^{At}x(0) \; + \int_0^t e^{A(t-\tau)}\, Bu(\tau)d\tau$

(convolution of e^{At} with $u(\tau)$)

- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau \qquad W_t \epsilon \mathbb{R}^{nxn}$
  - $W_t\xi = \lambda\xi$
  - $W_t \approx \mathbb{C}\mathbb{C}^T$

(^{T} = * if the values of A and B were complex)

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

```
>> rank(ctrb(A,b))
>> [U,S,V] = svd(ℂ, ‘econ’)
```

The SVD of A takes the form: $A = U\Sigma V^T$
U = left singular vector
V = right singular vector
Σ = diagonal matrix with singular values

Controllabillity Gramian

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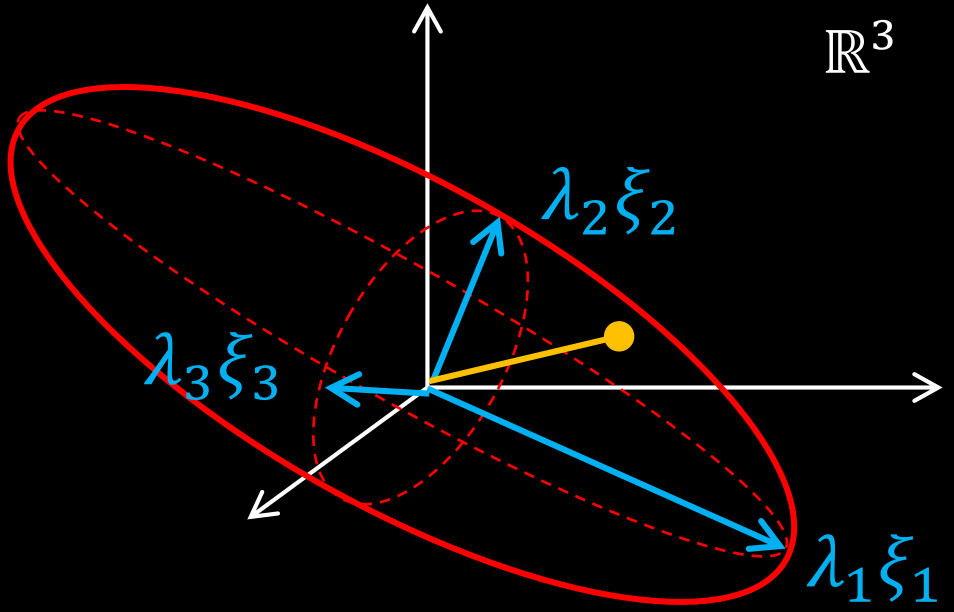

Controllabillity Gramian

By DLR, CC-BY 3.0, CC BY 3.0 de, https://commons.wikimedia.org/w/index.php?curid=61072555

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

```
>> rank(ctrb(A,b))
>> [U,S,V] = svd(ℂ, ‘econ’)
```

- Controllability for very high dimensional systems?
- Many directions in $\mathbb{R}^n$ are extremely stable - you only need to control directions that impact your control objective
- *Stabilizability*

Controllabillity Gramian

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$

(convolution of e^{At} with $u(\tau)$)

$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$

$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$

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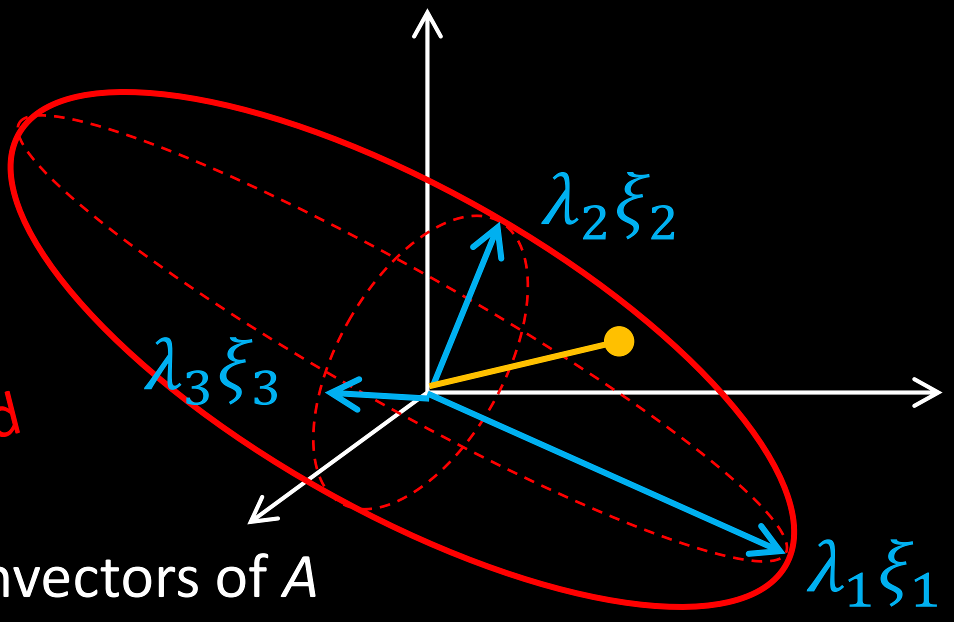
- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \quad W_t \epsilon \mathbb{R}^{nxn}$
  - $W_t\xi = \lambda\xi$
  - $W_t \approx \mathbb{C}\mathbb{C}^T$
- *Stabilizability*
  - A system is stabilizable iff all unstable eigenvectors of *A* are in the controllable subspace



lightly damped

PBH Test

PBH test

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

```
>> rank(ctrb(A,b))
```

- Popov-Belevitch-Hautus (PBH) test
  - (A,B) is ctrb iff (if A has *m* repeated eigenvalues, we need *m* actuation inputs)
    - $rank[(A - \lambda I) \quad B] = n \vee \quad \lambda \epsilon \mathbb{C}$
  - The (*A*,*B*) pair is controllable if and only if the rank of the concatenated matrix is n for all of the eigenvalues belonging to the complex plane!

1. $rank[(A - \lambda I)] = n$, except for at eigenvalues λ $A\xi = \lambda\xi$
2. *B* needs to have some component in each eigenvector direction
3. If *B* is a random vector (B=randn(n,1)), then (A,B) will be controllable with high probability.

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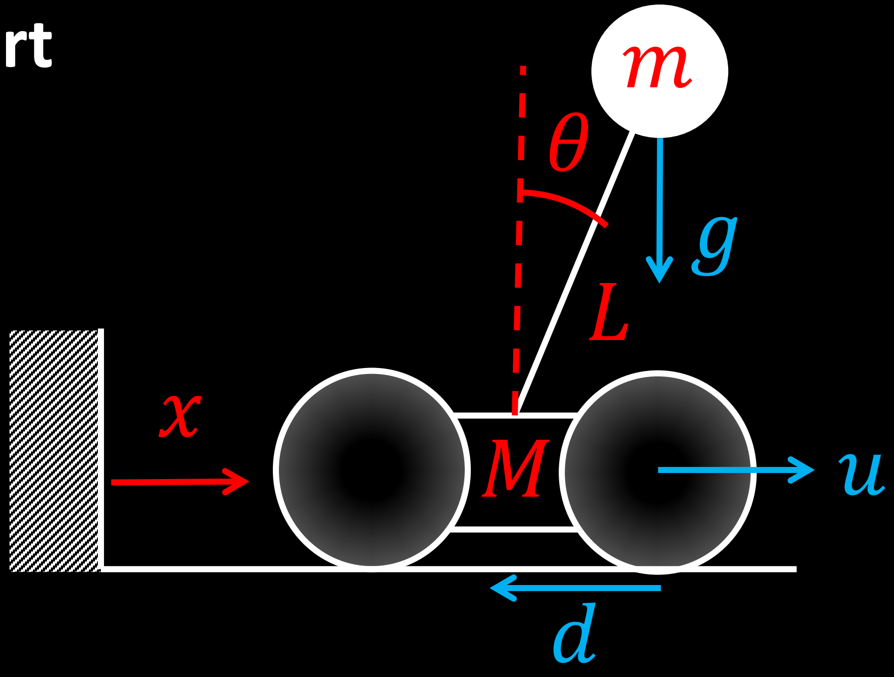
Inverted Pendulum on a Cart

Inverted Pendulum on a Cart



Force acting on the cart in the x direction

Eq. of motion

↓

State space model → Fixed points, $\bar{x}$ → Jacobian → (A,B) Controllable?

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\theta = \overset{\text{down}}{0}, \underset{\text{up}}{\pi}$$

$$\dot{\theta} = 0$$

$$\dot{x} = 0$$

x free variable

$$\frac{df}{dx}\Big|_{\bar{x}}$$

$$\dot{x} = Ax + Bu$$

↓

Add linear control

$$\dot{x} = (A - BK)x$$