ECE 4960

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Fast Robots

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Inverted pendulum dynamics
- LQR control

$$\dot{x} = Ax + Bu$$

This should look familiar from..
- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

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Inverted Pendulum on a Cart

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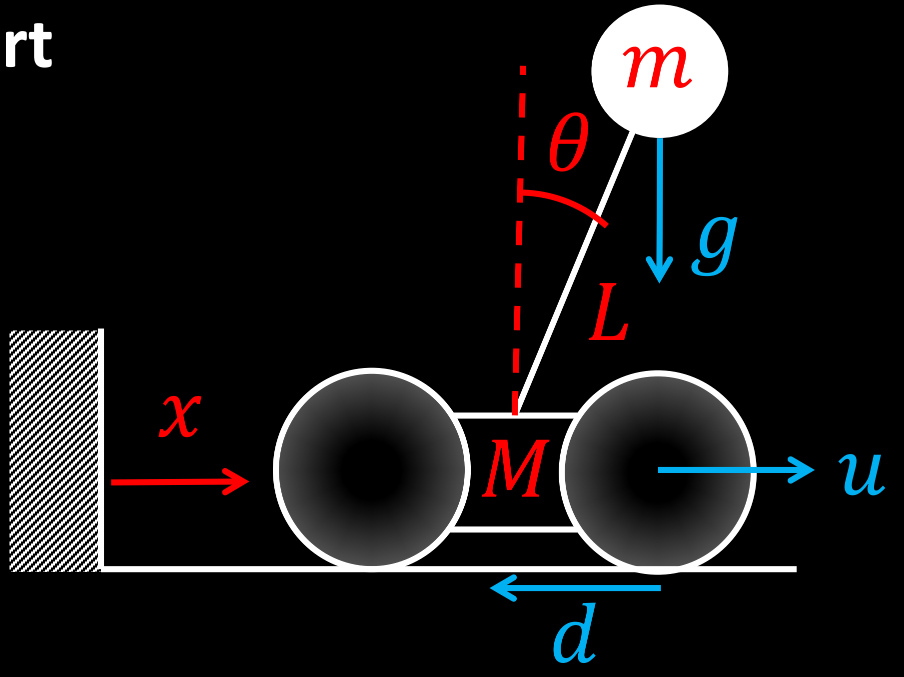


Force acting on the cart in the x direction

Eq. of motion

↓

State space model → Fixed points, $\bar{x}$ → Jacobian → (A,B) Controllable?

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

down

$$\theta = 0, \pi$$

up

$$\dot{\theta} = 0$$

$$\dot{x} = 0$$

x free variable

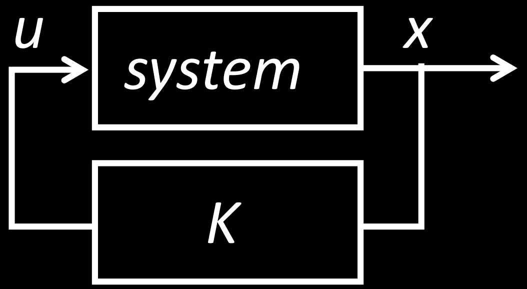
$$\frac{df}{dx}\Big|_{\bar{x}}$$

$$\dot{x} = Ax + Bu$$

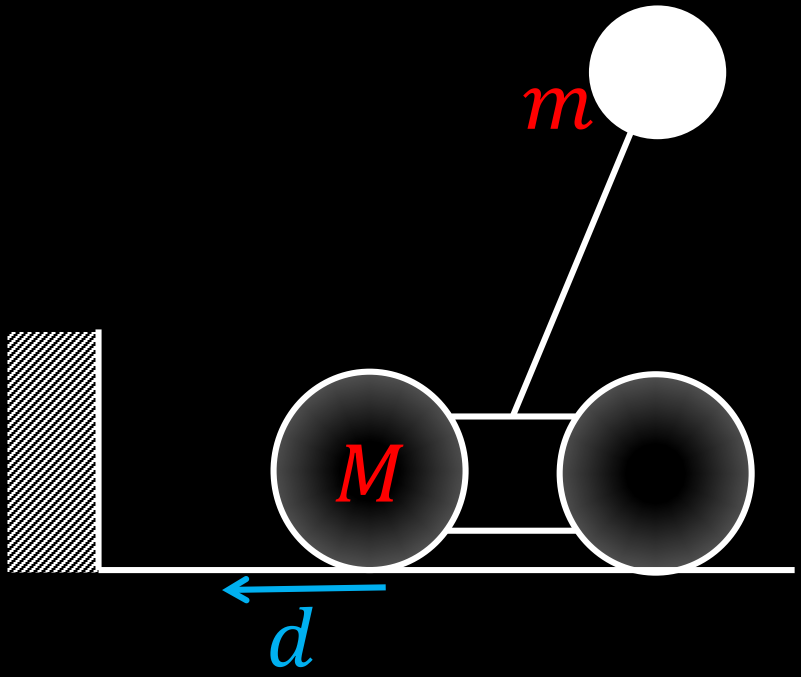
↓

Add linear control

$$\dot{x} = (A - BK)x$$

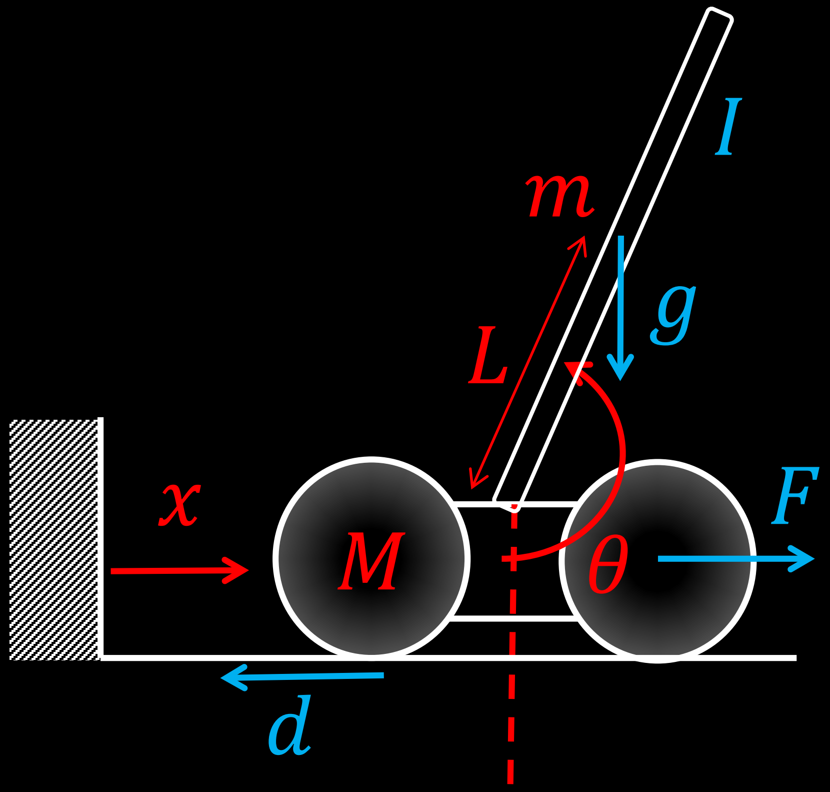


Inverted Pendulum on a Cart - Equations of Motion



M	Cart mass (0.475 kg)
d	Coefficient of friction for cart (1.8 Ns/m)
m	Pendulum mass

Inverted Pendulum on a Cart - Equations of Motion

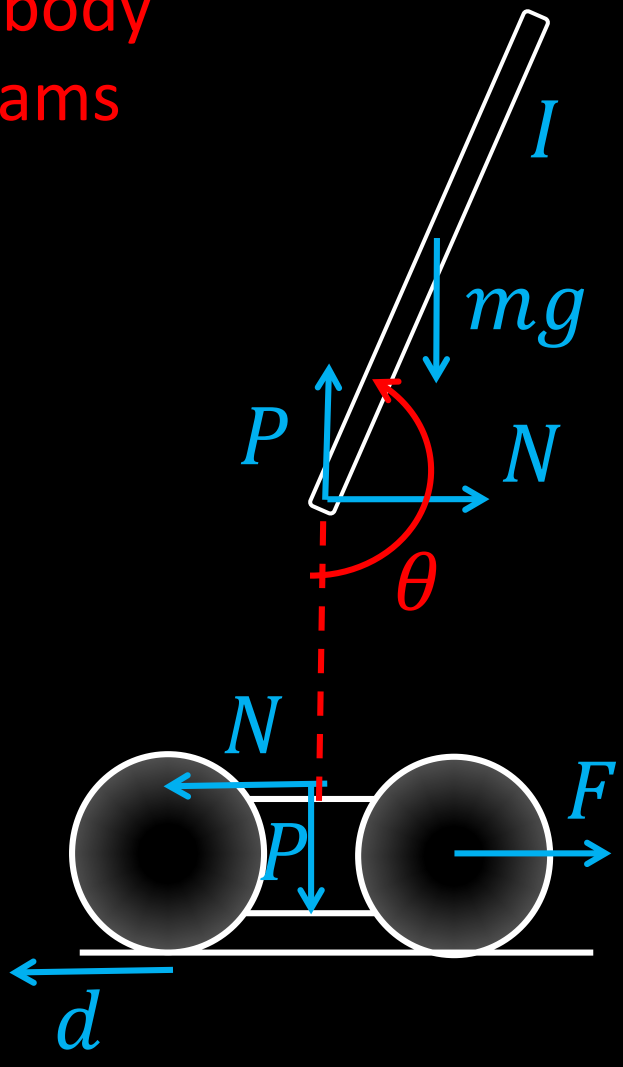


M	Cart mass (0.475 kg)
d	Coefficient of friction for cart (1.8 Ns/m)
m	Pendulum mass
L	Length to pendulum center of mass
I	Pendulum moment of inertia *
F	Force applied to the cart
x	Cart position coordinate
θ	Pendulum angle from vertical down

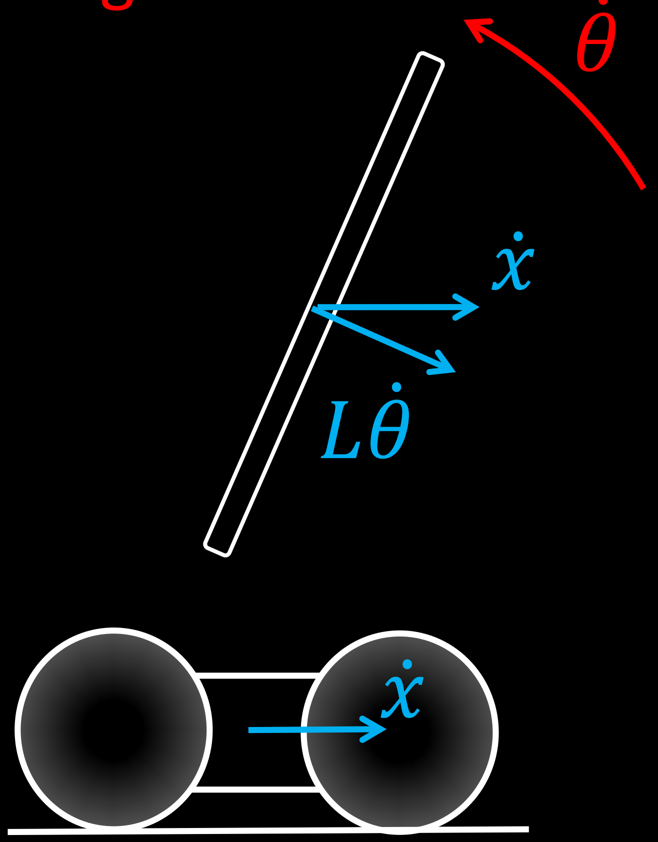
**…how accurate does the model have to be?*

Inverted Pendulum on a Cart – Three Basic Diagrams

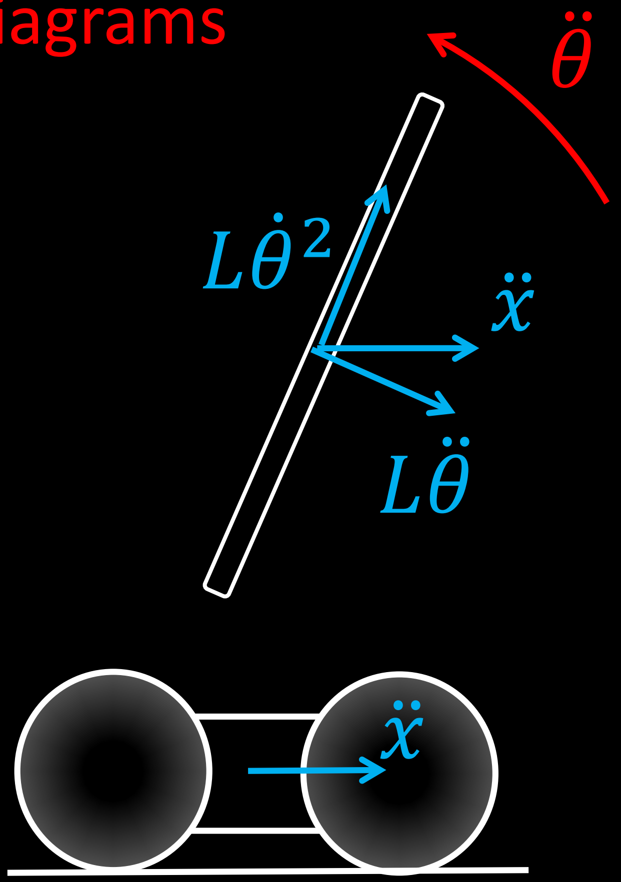
Free-body diagrams



Velocity diagrams

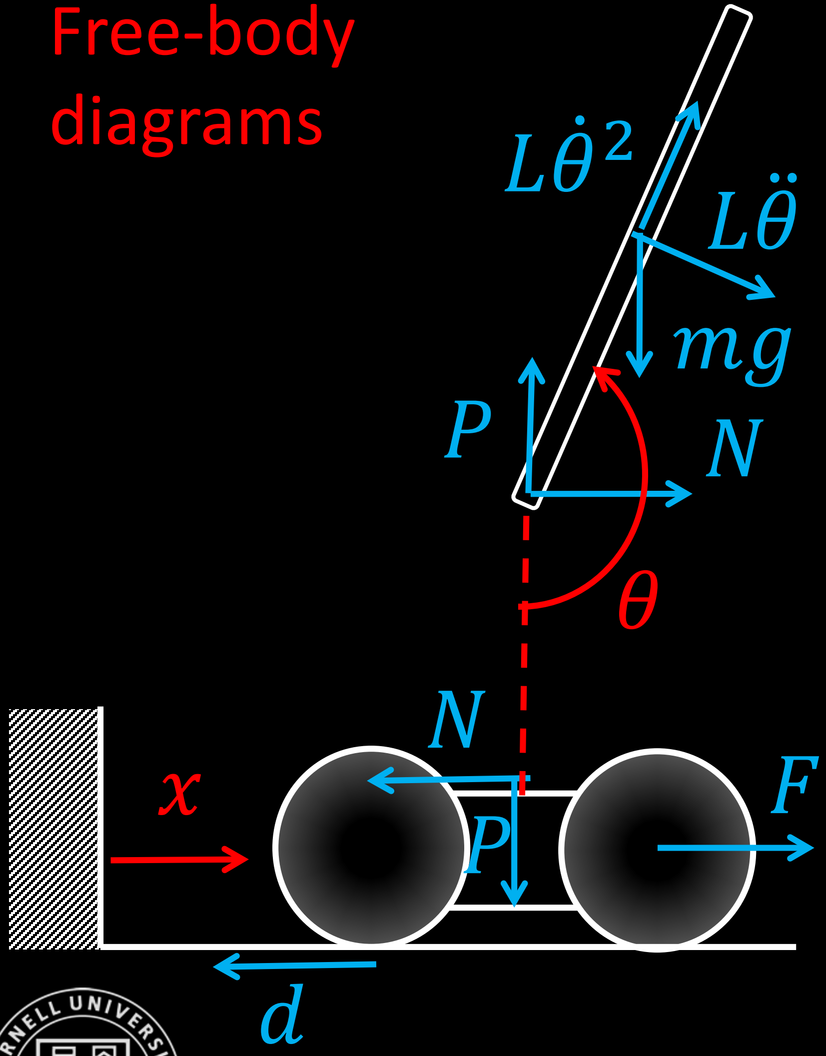


Acceleration diagrams



Inverted Pendulum on a Cart - Equations of Motion

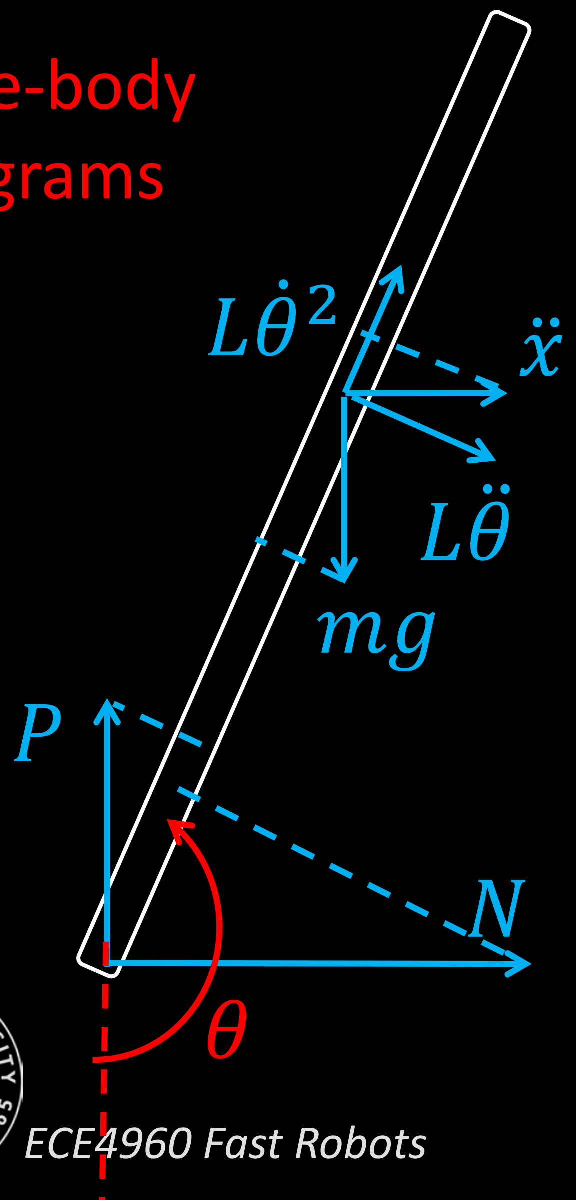
Free-body diagrams



- Sum the horizontal forces on the cart ($\Sigma F = ma$)
  - (1) $F - d\dot{x} - N = M\ddot{x}$
  - (2) $F = M\ddot{x} + d\dot{x} + N$
- Sum the horizontal forces on the pendulum
  - (3) $N + mL\dot{\theta}^2 sin\theta = m(\ddot{x} + L\ddot{\theta}cos\theta)$
  - (4) $N = m\ddot{x} + mL\ddot{\theta}cos\theta - mL\dot{\theta}^2 sin\theta$
- Substitute (4) into (2)
  - (5) $\boxed{F = (M + m)\ddot{x} + d\dot{x} + mL\ddot{\theta}cos\theta - mL\dot{\theta}^2 sin\theta}$

Inverted Pendulum on a Cart - Equations of Motion

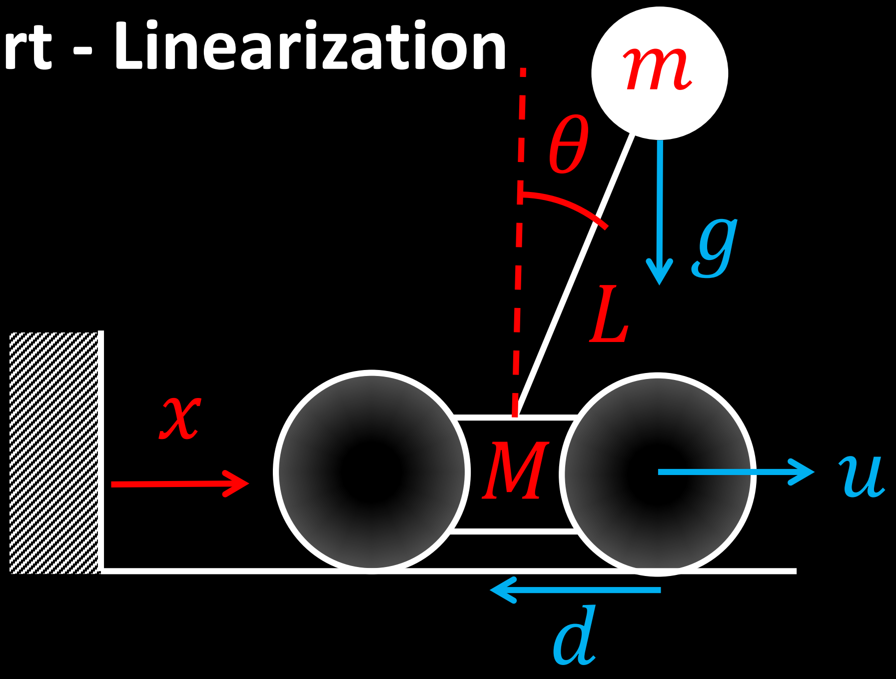
Free-body diagrams



- (5) $F = (M+m)\ddot{x} + d\dot{x} + mL\ddot{\theta}cos\theta - mL\dot{\theta}^2 sin\theta$

- Sum the forces perpendicular to the pendulum
  - (6) $(P - mg)sin\theta + Ncos\theta = m(L\ddot{\theta} + \ddot{x}cos\theta)$
- Sum the moments about the pendulum centroid
  - (7) $PLsin\theta + NLcos\theta = 0$ $(-I\ddot{\theta})$
- Combine (6) and (7)
  - (8) $mL\ddot{\theta} + mgsin\theta = -m\ddot{x}cos\theta$
  - (9) $L\ddot{\theta} + gsin\theta = -\ddot{x}cos\theta$

Inverted Pendulum on a Cart - Linearization



Force acting on the cart in the x direction

Eq. of motion

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State space model → Fixed points, $\bar{x}$ → Jacobian → (A,B) Controllable?

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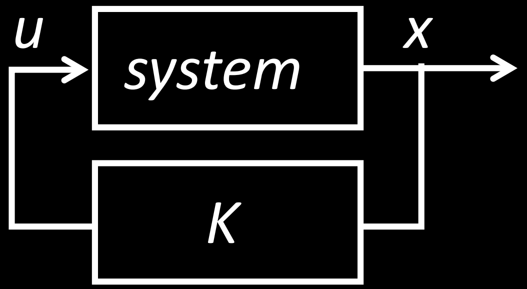
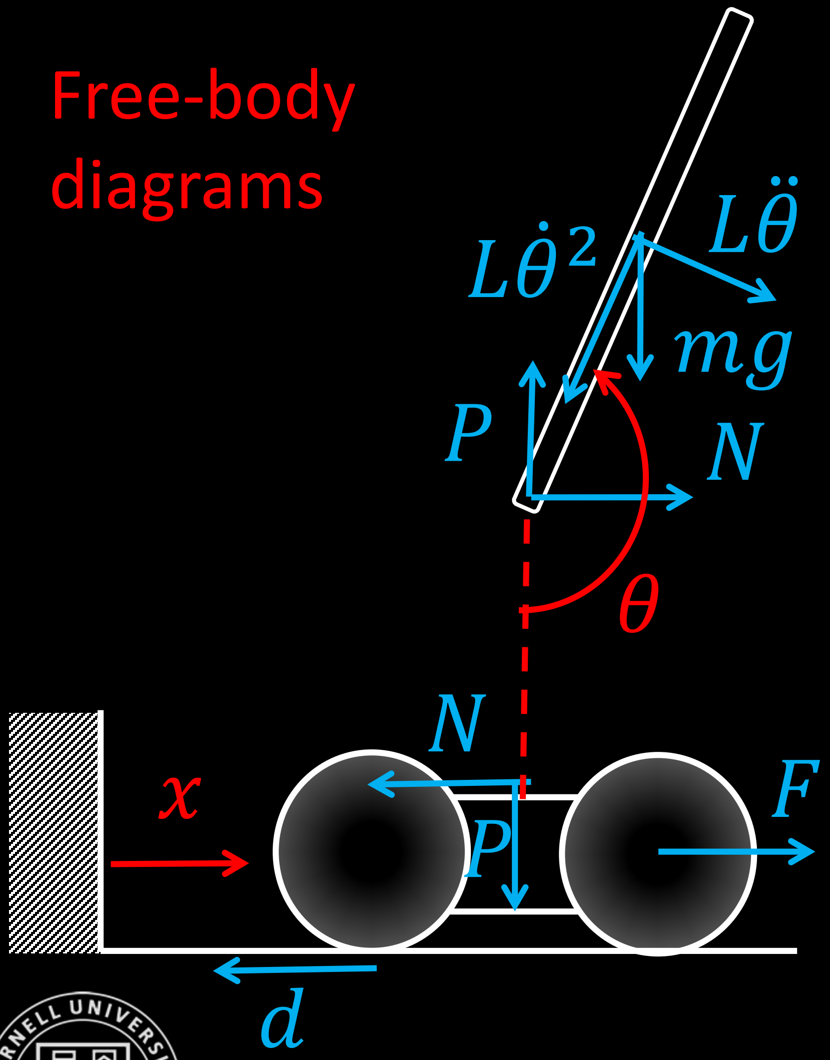


Inverted Pendulum on a Cart - Linearization

Free-body diagrams



Equations of motion

- (5) $F = (M+m)\ddot{x} + d\dot{x} + mL\ddot{\theta}cos\theta - mL\dot{\theta}^2 sin\theta$
- (9) $L\ddot{\theta} + gsin\theta = -\ddot{x}cos\theta$

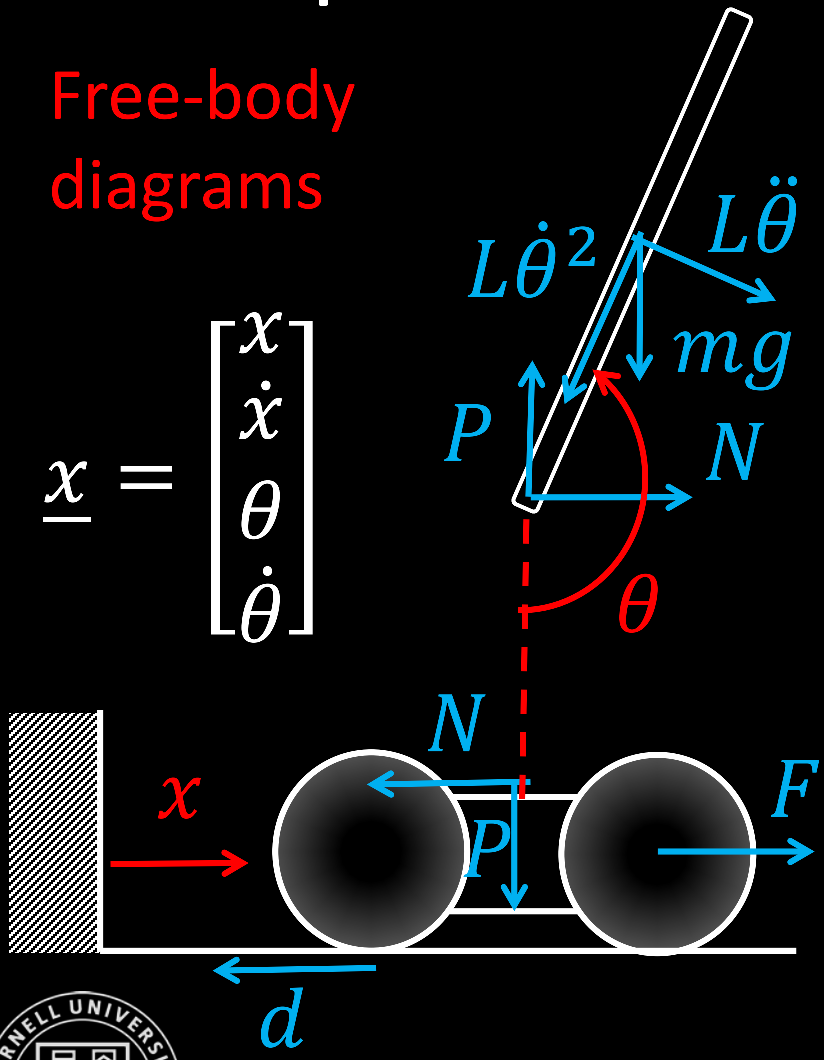
- Fixed points: $\theta = [0, \pi], \dot{\theta} = 0, \dot{x} = 0$
- Jacobian
  - $\dot{\theta}^2 = \dot{\varphi}^2 \approx 0$
  - small angle approximation
    - $cos\theta = \cos(\pi + \varphi) \approx -1$
    - $sin\theta = \sin(\pi + \varphi) \approx -\varphi$

Linearized about $\theta = \pi$

- (10) $F = u = (M+m)\ddot{x} + d\dot{x} - mL\ddot{\varphi}$
- (11) $L\ddot{\varphi} - g\varphi = \ddot{x}$

Inverted Pendulum on a Cart – State Space

Free-body diagrams



$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

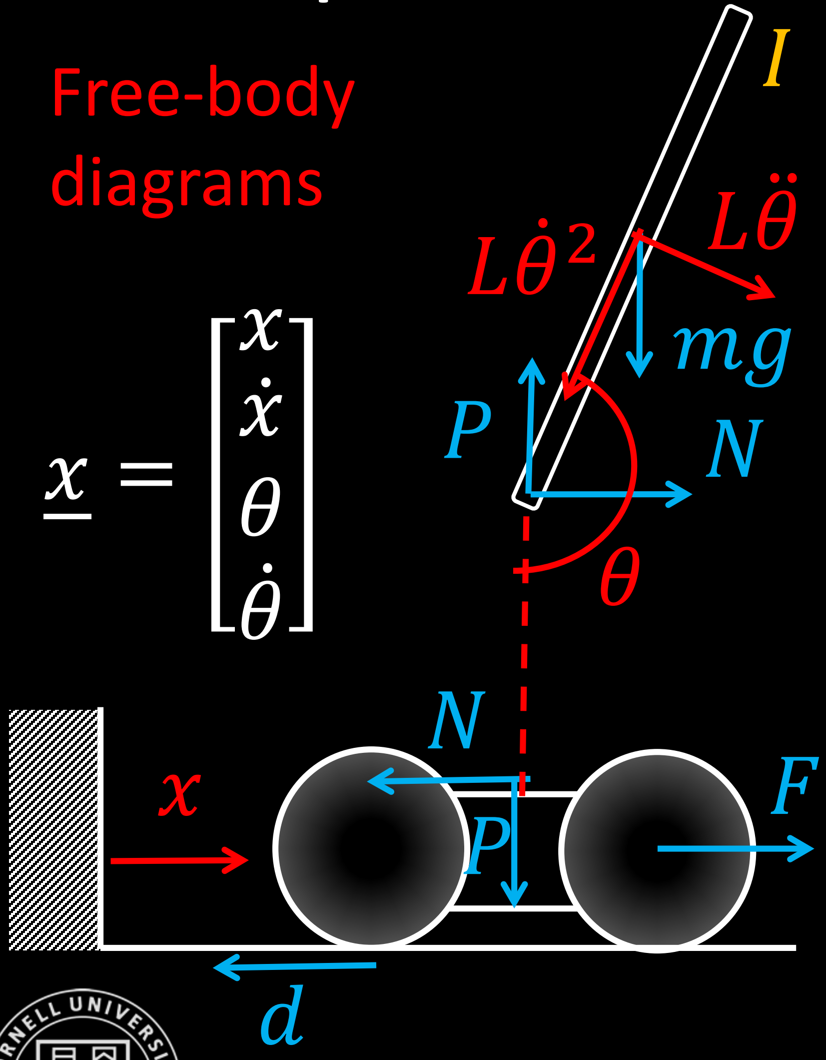
Linearized about $\theta = \pi$

- (10) $F = u = (M+m)\ddot{x} + d\dot{x} - mL\ddot{\varphi}$
- (11) $L\ddot{\varphi} - g\varphi = \ddot{x}$

- Substitute (11) into (10)
  - (12) $(M+m)(\ddot{\varphi}L - g\varphi) + d\dot{x} - mL\ddot{\varphi} = u$
  - (13) $M\ddot{\varphi}L - (M+m)g\varphi + d\dot{x} = u$
  - (14) $\ddot{\varphi} = \frac{M+m}{ML}g\varphi - \frac{d}{ML}\dot{x} + \frac{1}{ML}u$
- Plug (14) back into (11)
  - (15) $\ddot{x} = \frac{(M+m)g}{M}\varphi - \frac{d}{M}\dot{x} + \frac{1}{M}u - g\varphi$
  - (16) $\ddot{x} = \frac{m}{M}g\varphi - \frac{d}{M}\dot{x} + \frac{1}{M}u$

Inverted Pendulum on a Cart – State Space

Free-body diagrams



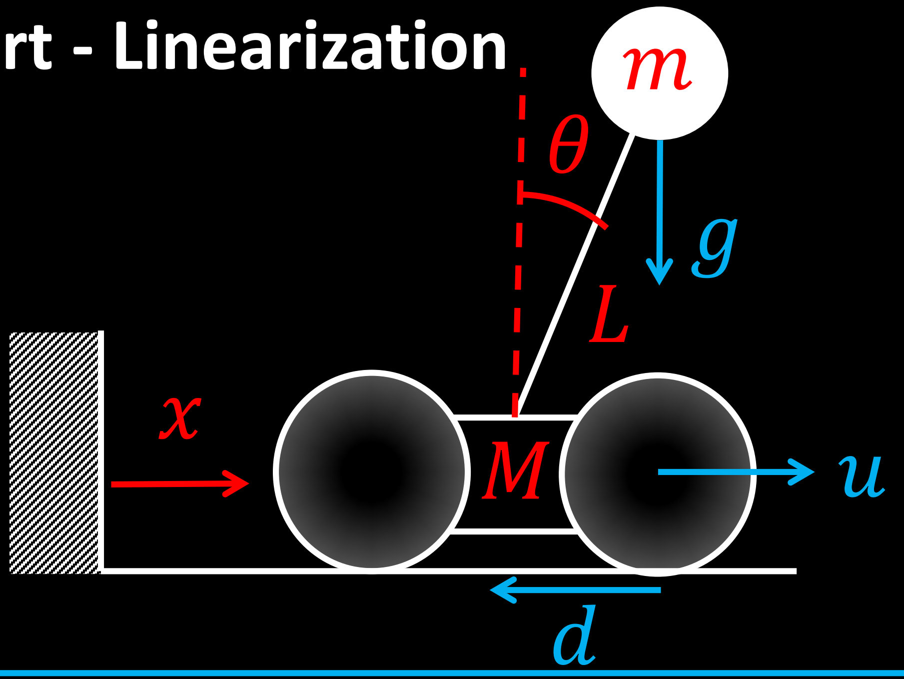
$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

Linearized about $\theta = \pi$

- (14) $\ddot{\varphi} = \frac{(M+m)g}{ML}\varphi - \frac{d}{ML}\dot{x} + \frac{1}{ML}u$
- (16) $\ddot{x} = \frac{m}{M}g\varphi - \frac{d}{M}\dot{x} + \frac{1}{M}u$
- $\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{M} & \frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{d}{ML} & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(I+ml^2)b}{I(M+m)+Mml^2} \\ 0 \\ \frac{-ml}{I(M+m)+Mml^2} \end{bmatrix} \mathrm{u}$$

Inverted Pendulum on a Cart - Linearization



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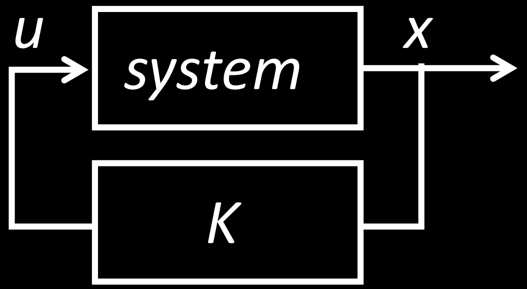

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x free variable

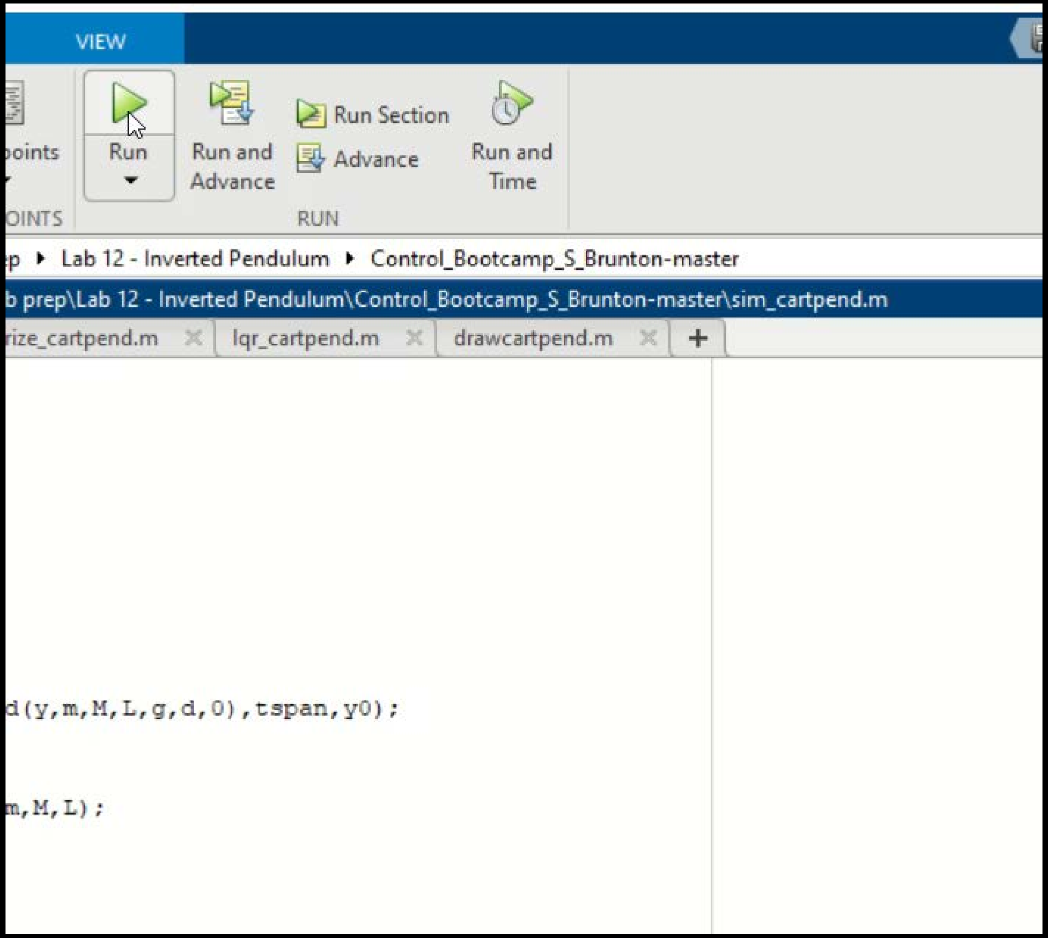
$$\frac{df}{dx}|_{\bar{x}}$$

$$\dot{x} = Ax + Bu$$

Inverted Pendulum on a Cart - Linearization

Matlab example

- Non-linear model
- Linearized model
- Eigenvalues
- Stability
- Controllability



p ▸ Lab 12 - Inverted Pendulum ▸ Control_Bootcamp_S_Brunton-master

b prep\Lab 12 - Inverted Pendulum\Control_Bootcamp_S_Brunton-master\sim_cartpend.m

```
d(y,m,M,L,g,d,0),tspan,y0);


m,M,L);
```

$\gg eig(A)$

$\lambda_4 = 3.5069 \lambda_3 = -1.9$

$\gg rank(ctrb(A,B))$

4

Inverted Pendulum on a Cart - Linearization

Matlab example

- Non-linear model
- Linearized model
- Eigenvalues
- Stability
- Controllability
- Add control

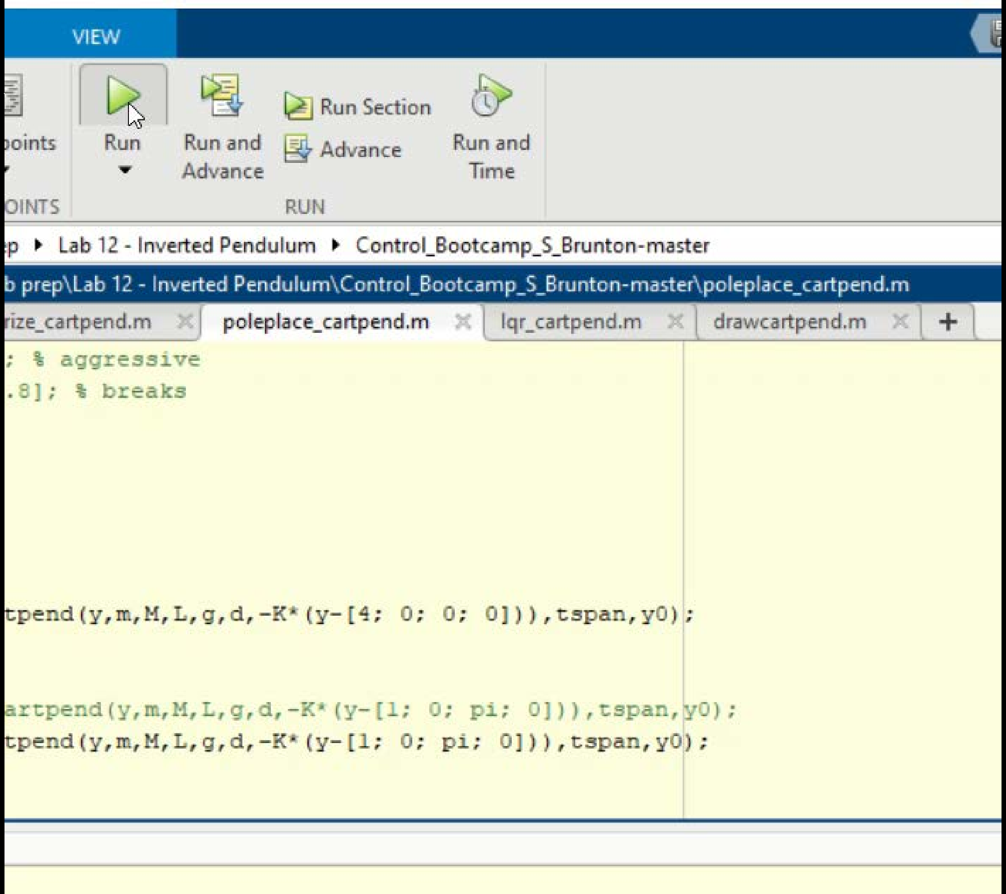
$\gg eigs = [-1.1; -1.2; -1.3; -1.4]$

$\gg K = place(A,B,eigs)$

$K=[-0.0965 \quad -1.3111 \quad 8.7254 \quad 2.2295]$

$\gg eig(A-B.*K)$

$[-1.4; -1.3; -1.2; -1.$



```
; % aggressive
.8]; % breaks




tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);


artpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
tpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
```

Pole Placement

- In Python
  - https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.place_poles.html
  - K = scipy.signal.place_poles(A, B, poles)
- Barely stable eigenvalues
  - Not enough control authority
- More negative eigenvalues
  - Faster dynamics
  - Less robust system
- Linear Quadratic Control (LQR)
  - “Sweet spot of eigenvalues”
  - Balances how fast you stabilize your state and how much control energy you spend to get there

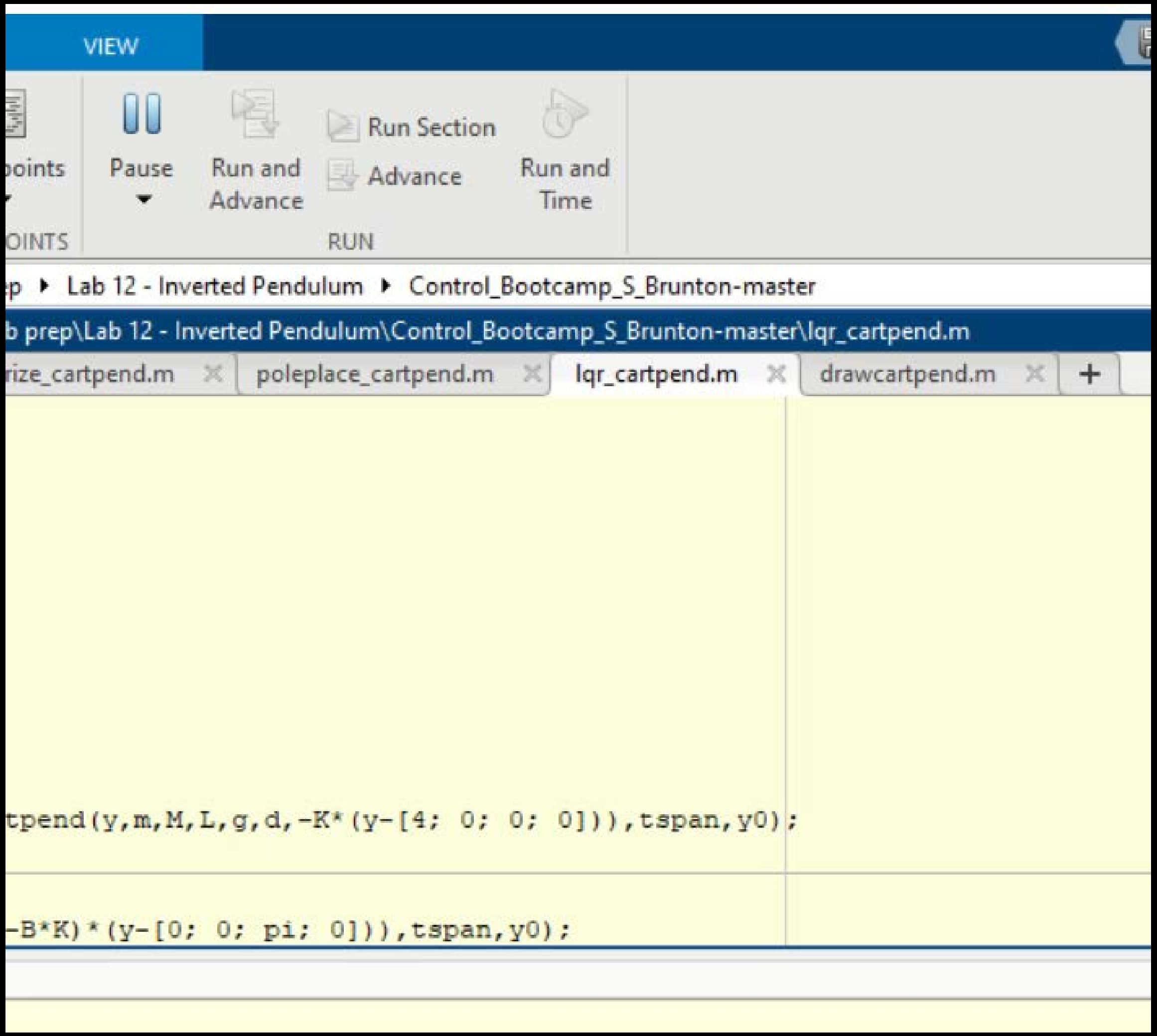
Linear Quadratic Control

$$\dot{x} = Ax + Bu, x \epsilon \mathbb{R}^n$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$

- >> K = place(A,B,eigs)
- Where are the best eigs??
  - Linear Quadratic Regulator (LQR)
    - >> K = lqr(A,B,Q,R)
    - $\int_0^\infty (x^T Q x + u^T R u) dt$
    - $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
    - Ricotta equation
      - Computationally expensive, O(n^3)

Matlab Example

- $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
- K = lqr(A,B,Q,R);
- >>[T,D] = eigs(A-B.*K)
  - λ_1= -788.29 + 0.00i
  - λ_2= -0.70 + 0.83i
  - λ_3=-0.70 - 0.83i
  - λ_4=-0.83 + 0.00i
- T(:,1)
  - = $[0.0008, -0.6387, 0.0010, -0.7695]^T$



```
tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);

-B*K)*(y-[0; 0; pi; 0])),tspan,y0);
```

Matlab Example

- $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
- K = lqr(A,B,Q,R);
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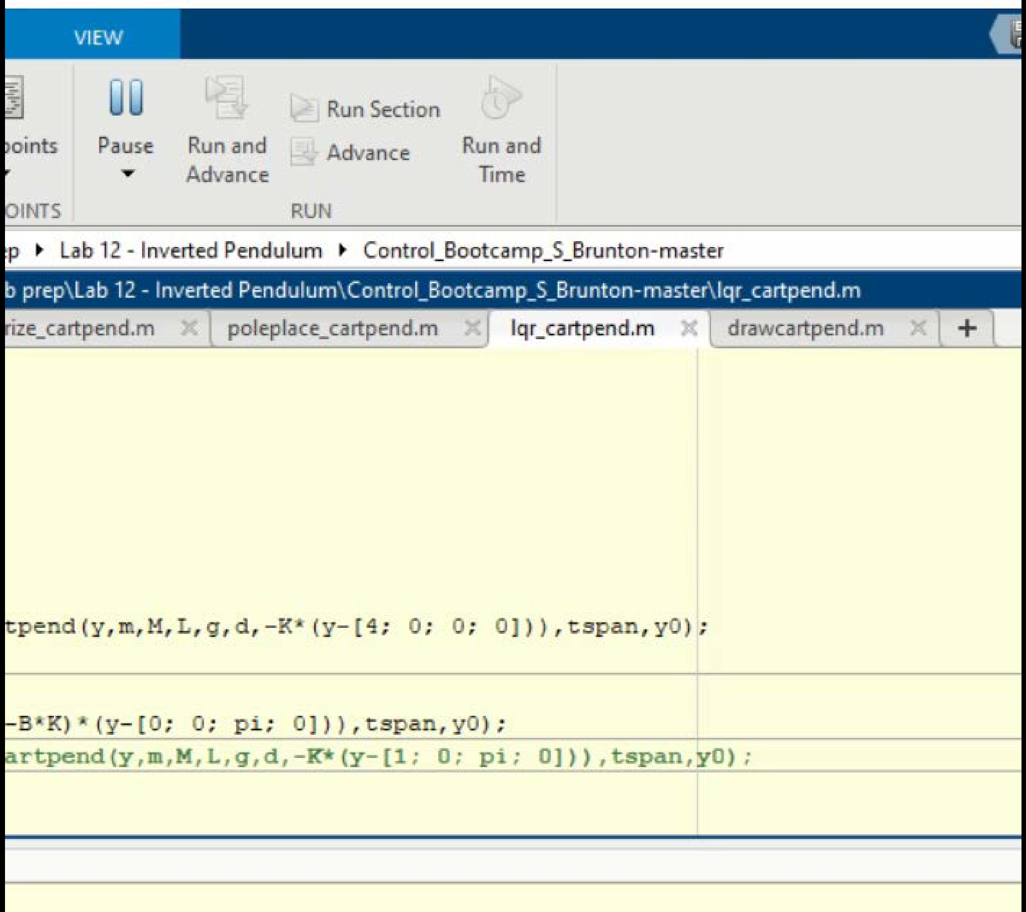
    - λ_3=-0.70 - 0.83i
    - λ_4=-0.83 + 0.00i
- T(:,1)
    - = [0.0008, -0.6387, 0.0010, -0.7695]^T

$R = 1$

λ_1 = -25.6851 + 0.0000i
λ_2 = -1.0855 + 0.8921i
λ_3 = -1.0855 - 0.8921i
λ_4 = -0.4811 + 0.0000i



```
tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);

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```

Linear Quadratic Control

$$\dot{x} = Ax + Bu, x\epsilon\mathbb{R}^n$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$

- >> K = place(A,B,eigs)
- Where are the best eigs??
  - Linear Quadratic Regulator (LQR)
    - >> K = lqr(A,B,Q,R)
    - $\int_0^\infty (x^T Qx + u^T Ru)dt$
    - $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.00$
    - Riccati equation
      - Computationally expensive, O(n^3)

- *The linear controller works!*
  - *(in simulation)*
- *Issues in Practice?*
  - *Imperfect models*
  - *Nonlinear parts*
    - *Deadband, saturation, etc.*
  - *We don’t have full state feedback*

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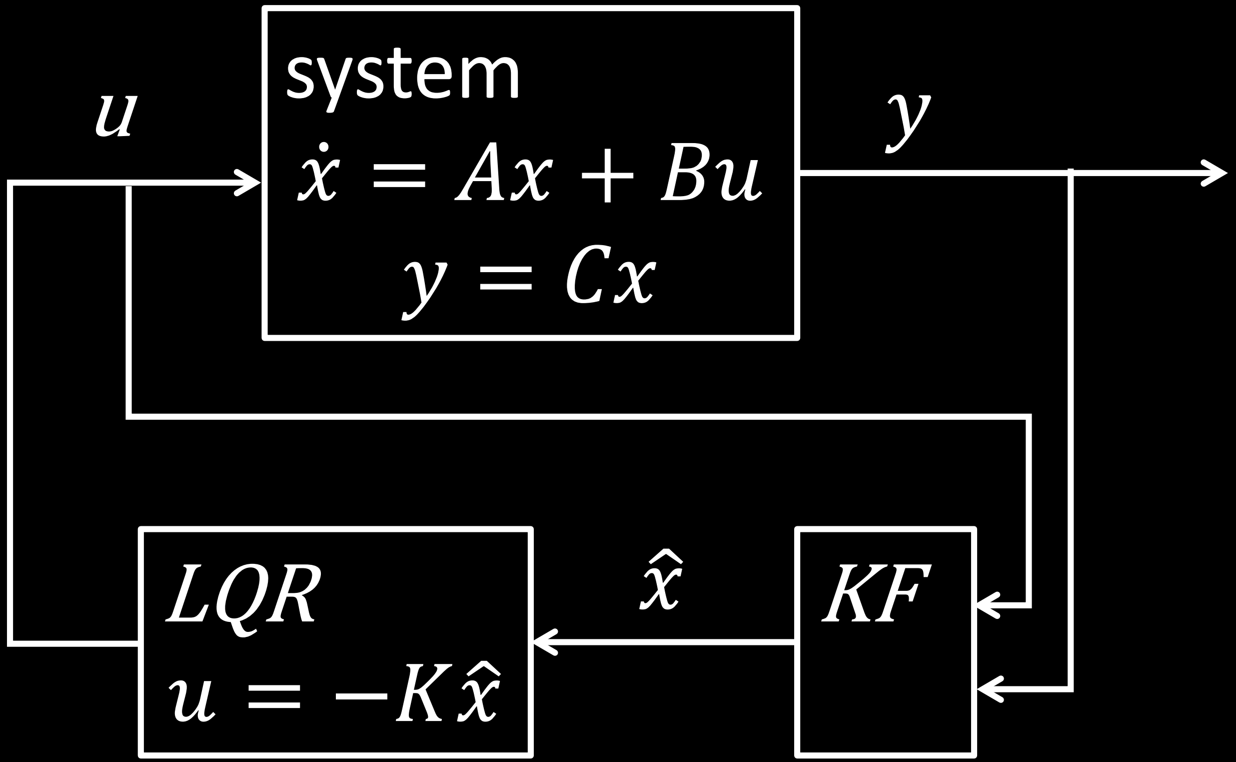
Full State Feedback

Full State Feedback

- Controllability
  - Can we steer the system anywhere given some control input u?
- Observability
  - Can we estimate any state x, from a time series of your measurements y(t)?

$$\dot{x} = Ax + Bu, x \epsilon \mathbb{R}^n$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$



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Any last questions on Lab 10?