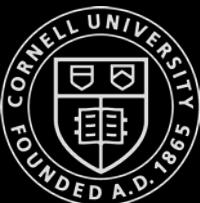


ECE 4960

Prof. Kirstin Hagelskjær Petersen
kirstin@cornell.edu

Fast Robots



ECE4960 Fast Robots

Linear Systems

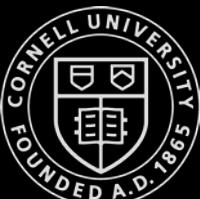
- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Inverted pendulum dynamics
- LQR control

Based on “Control Bootcamp”, Steve Brunton, UW
<https://www.youtube.com/watch?v=Pi7l8mMjYVE>

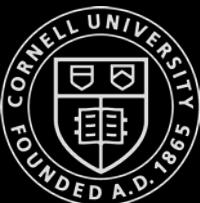
$$\dot{x} = Ax + Bu$$

This should look familiar from..

- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc...



Inverted Pendulum on a Cart



Inverted Pendulum on a Cart

Eq. of motion

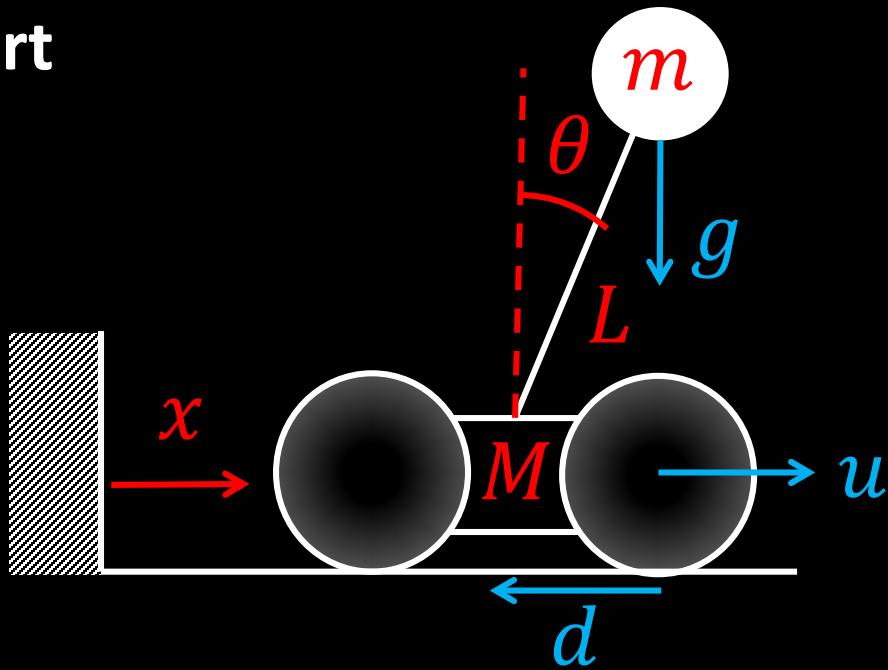


State space
model

→ Fixed points, \bar{x} → Jacobian → (A,B) Controllable?

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\begin{aligned} \theta &= 0, \pi \quad \text{down} \\ \dot{\theta} &= 0 \quad \text{up} \\ \dot{x} &= 0 \\ x &\text{ free variable} \end{aligned}$$

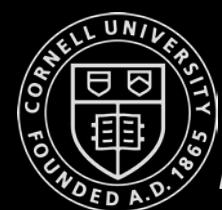
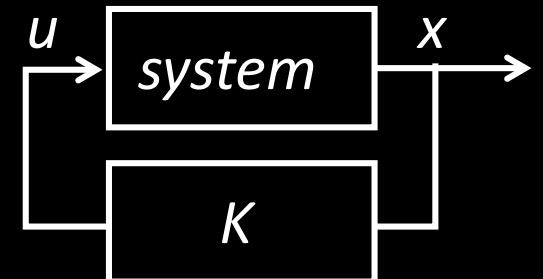


Force acting on the
cart in the x direction

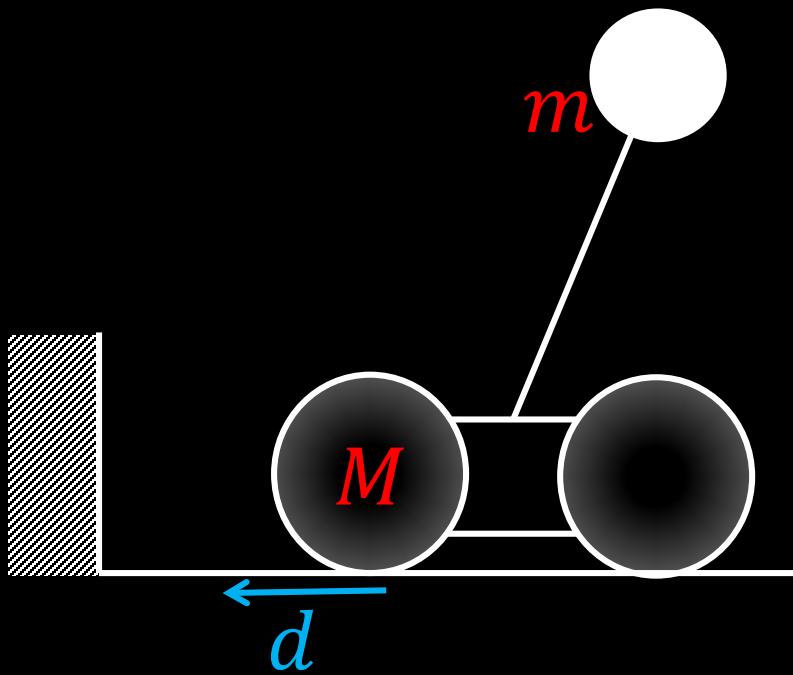
$$\frac{df}{dx} \Big|_{\bar{x}}$$

$$\dot{x} = Ax + Bu$$

↓
Add linear control
 $\dot{x} = (A - BK)x$



Inverted Pendulum on a Cart - Equations of Motion



M

d

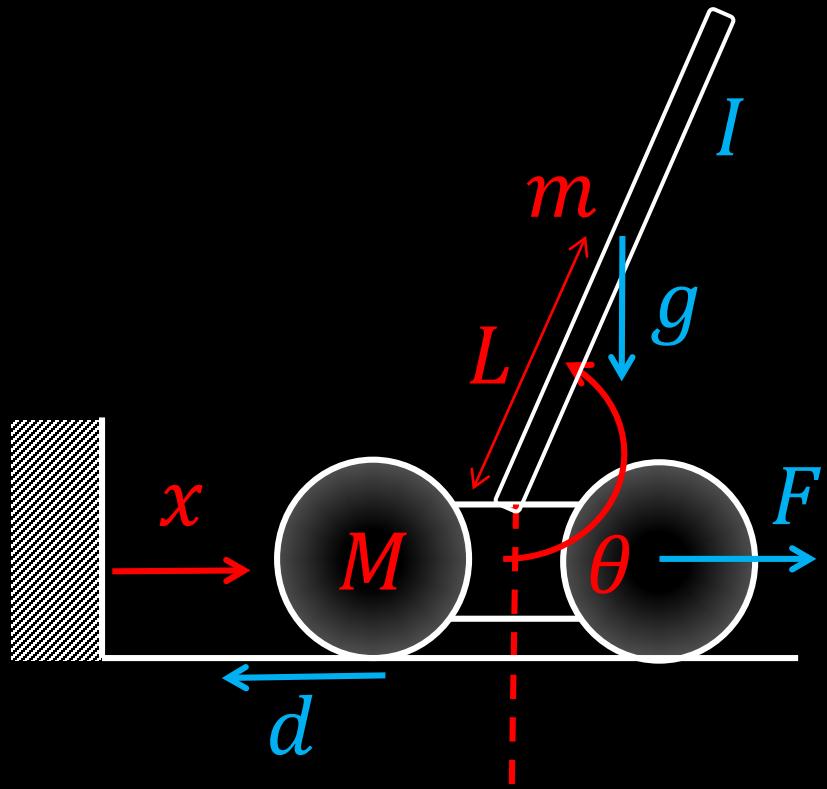
m

Cart mass (0.475 kg)

Coefficient of friction for cart (1.8 Ns/m)

Pendulum mass

Inverted Pendulum on a Cart - Equations of Motion

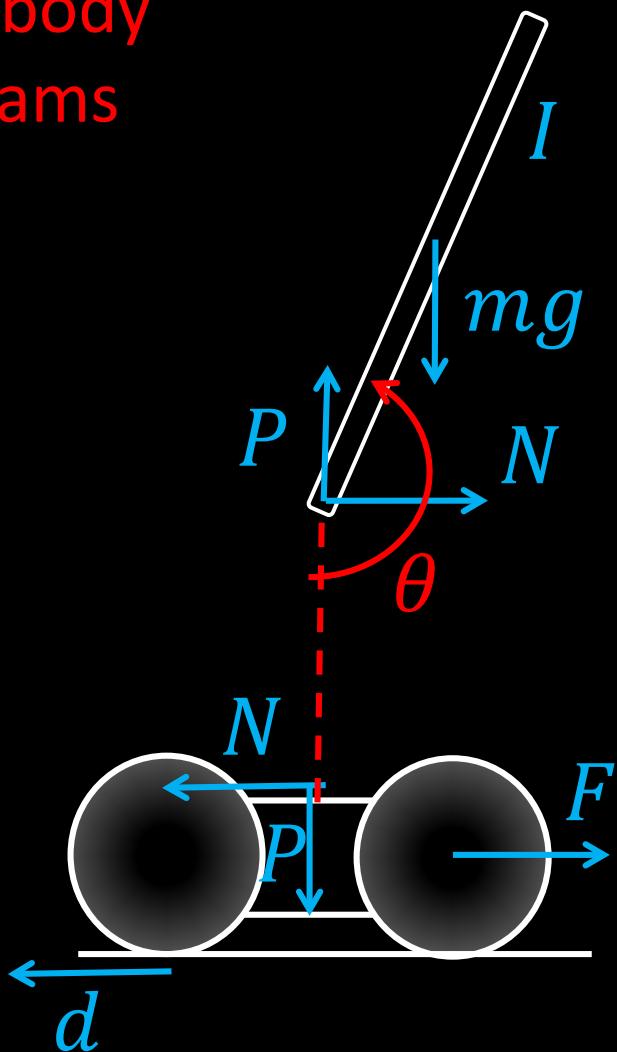


M	Cart mass (0.475 kg)
d	Coefficient of friction for cart (1.8 Ns/m)
m	Pendulum mass
L	Length to pendulum center of mass
I	Pendulum moment of inertia *
F	Force applied to the cart
x	Cart position coordinate
θ	Pendulum angle from vertical down

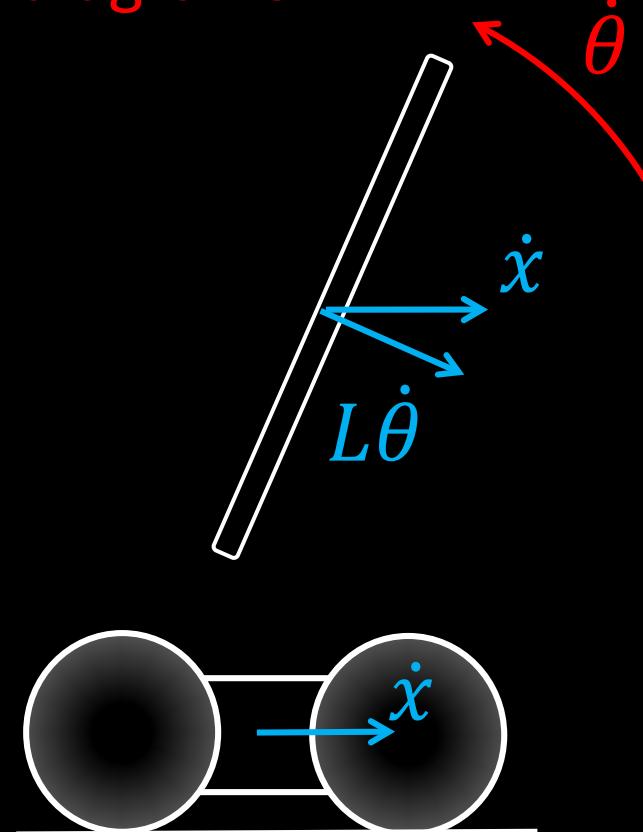
**...how accurate does the model have to be?*

Inverted Pendulum on a Cart – Three Basic Diagrams

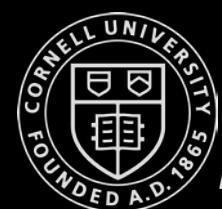
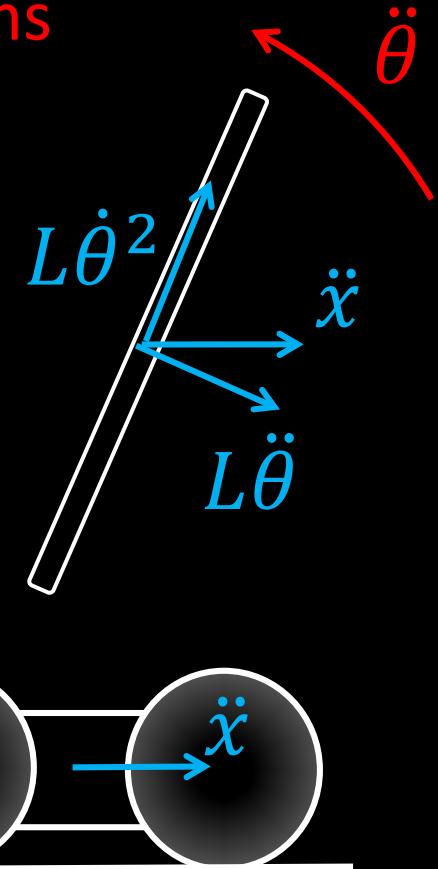
Free-body diagrams



Velocity diagrams

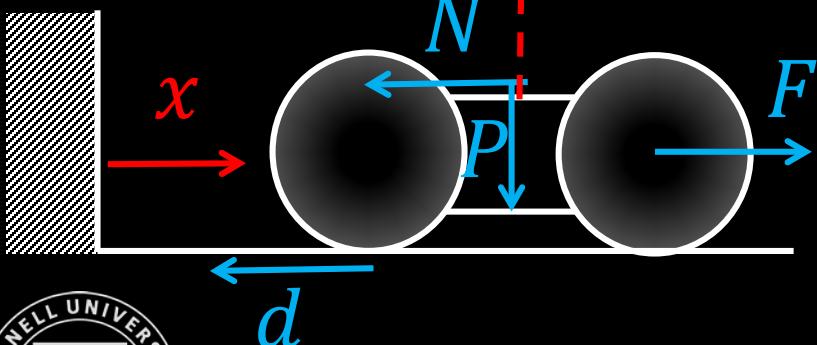
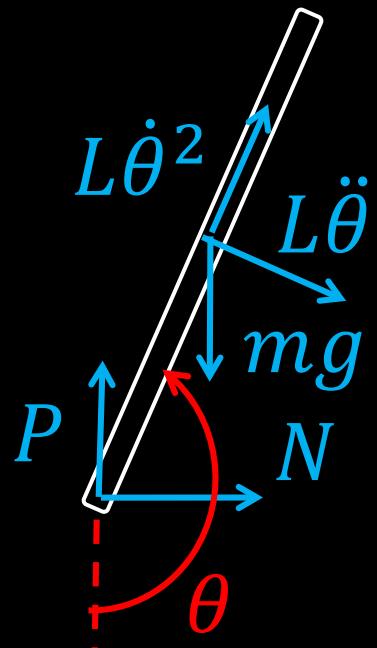


Acceleration diagrams

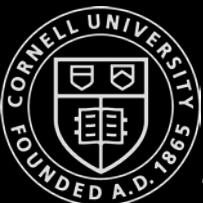


Inverted Pendulum on a Cart - Equations of Motion

Free-body
diagrams

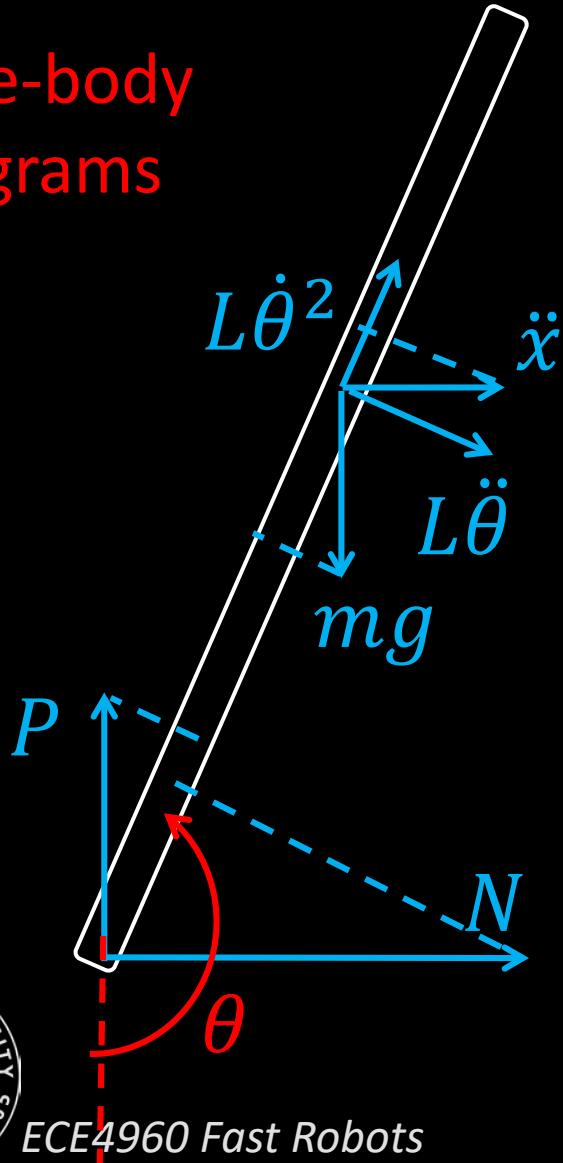


- Sum the horizontal forces on the cart ($\Sigma F = ma$)
 - (1) $F - d\dot{x} - N = M\ddot{x}$
 - (2) $F = M\ddot{x} + d\dot{x} + N$
- Sum the horizontal forces on the pendulum
 - (3) $N + mL\dot{\theta}^2 \sin\theta = m(\ddot{x} + L\ddot{\theta} \cos\theta)$
 - (4) $N = m\ddot{x} + mL\ddot{\theta} \cos\theta - mL\dot{\theta}^2 \sin\theta$
- Substitute (4) into (2)
 - (5) $F = (M + m)\ddot{x} + d\dot{x} + mL\ddot{\theta} \cos\theta - mL\dot{\theta}^2 \sin\theta$



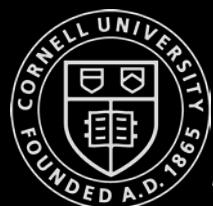
Inverted Pendulum on a Cart - Equations of Motion

Free-body
diagrams



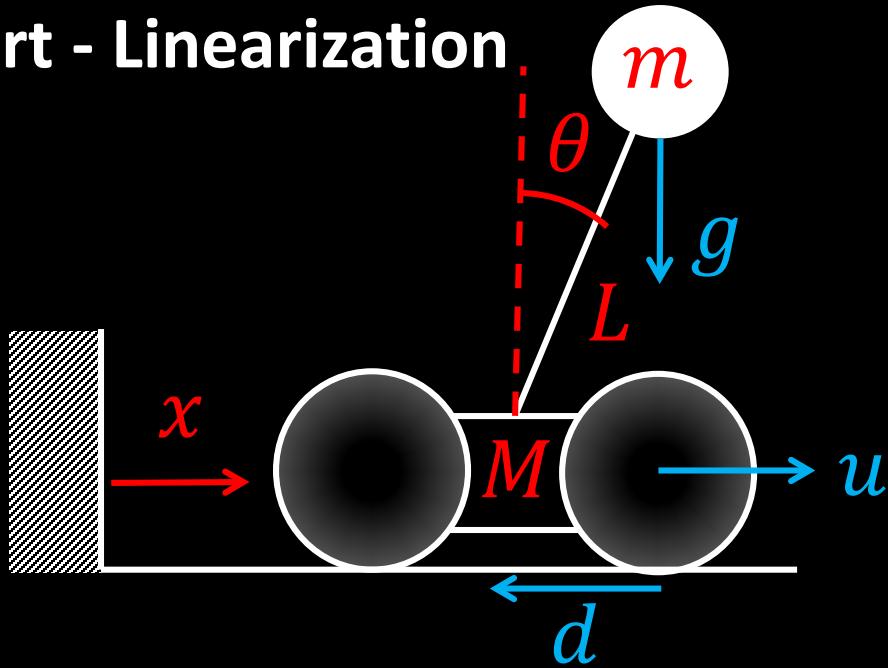
- (5) $F = (M + m)\ddot{x} + d\dot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta$

- Sum the forces perpendicular to the pendulum
 - (6) $(P - mg)\sin\theta + N\cos\theta = m(L\ddot{\theta} + \ddot{x}\cos\theta)$
- Sum the moments about the pendulum centroid
 - (7) $PL\sin\theta + NL\cos\theta = 0 \quad (-I\ddot{\theta})$
- Combine (6) and (7)
 - (8) $mL\ddot{\theta} + mgs\in\theta = -m\ddot{x}\cos\theta$
 - (9) $L\ddot{\theta} + g\sin\theta = -\ddot{x}\cos\theta$



Inverted Pendulum on a Cart - Linearization

Eq. of motion



Force acting on the cart in the x direction

State space model

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\begin{aligned} \theta &= 0, \pi \quad \text{down} \\ \dot{\theta} &= 0 \quad \text{up} \\ \dot{x} &= 0 \\ x &\text{ free variable} \end{aligned}$$

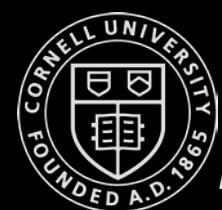
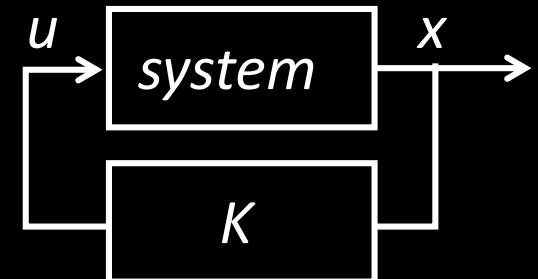
Jacobian

$$\frac{df}{dx} \Big|_{\bar{x}}$$

$$\dot{x} = Ax + Bu$$

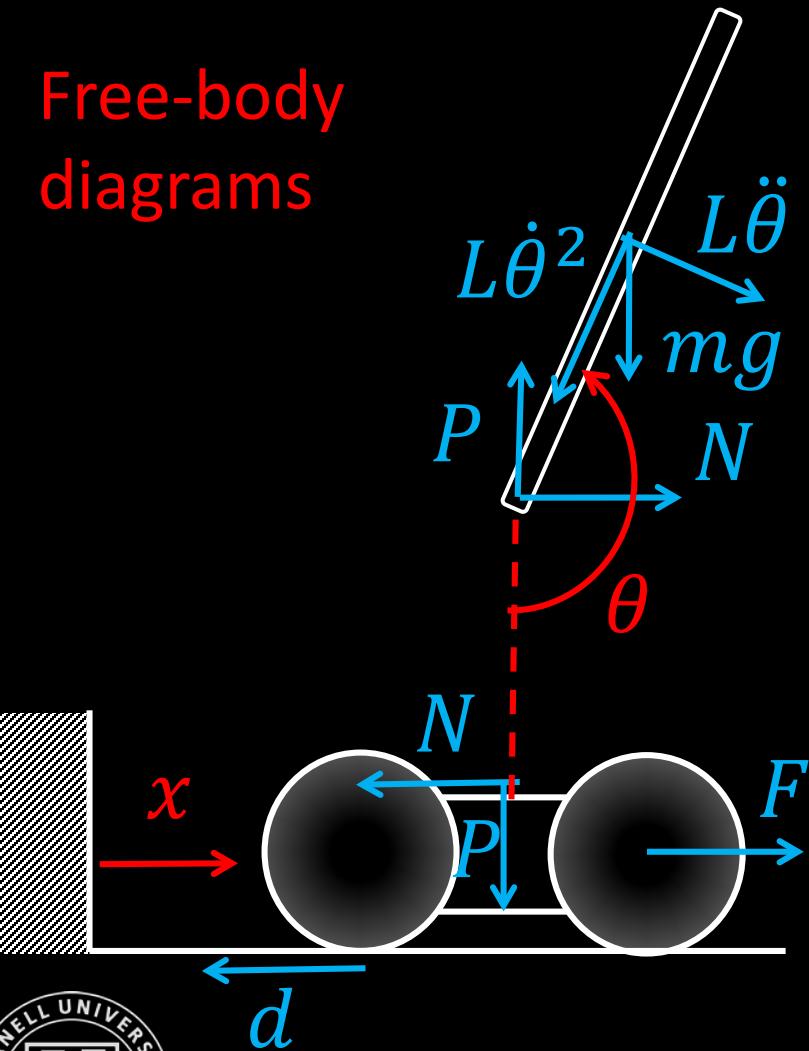
(A,B) Controllable?

$$\begin{aligned} &\downarrow \\ \text{Add linear control} \\ \dot{x} &= (A - BK)x \end{aligned}$$



Inverted Pendulum on a Cart - Linearization

Free-body
diagrams



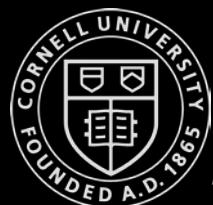
Equations of motion

- (5) $F = (M + m)\ddot{x} + d\dot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta$
- (9) $L\ddot{\theta} + g\sin\theta = -\ddot{x}\cos\theta$

- Fixed points: $\theta = [0, \pi], \dot{\theta} = 0, \ddot{x} = 0$
- Jacobian
 - $\dot{\theta}^2 = \dot{\varphi}^2 \approx 0$
 - small angle approximation
 - $\cos\theta = \cos(\pi + \varphi) \approx -1$
 - $\sin\theta = \sin(\pi + \varphi) \approx -\varphi$

Linearized about $\theta = \pi$

- (10) $F = u = (M + m)\ddot{x} + d\dot{x} - mL\dot{\varphi}$
- (11) $L\ddot{\varphi} - g\varphi = \ddot{x}$

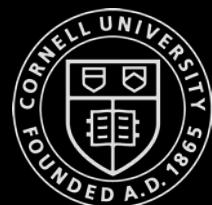
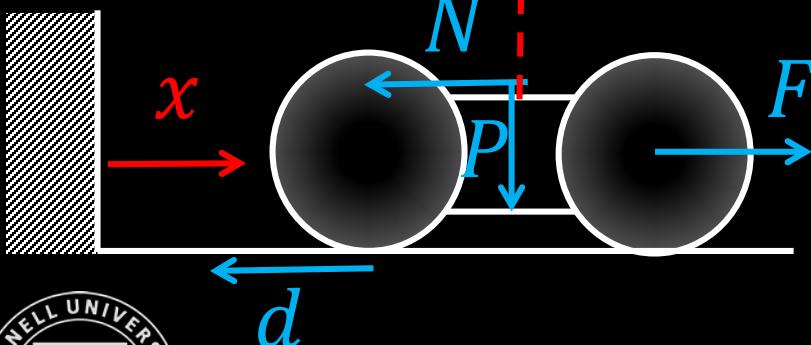
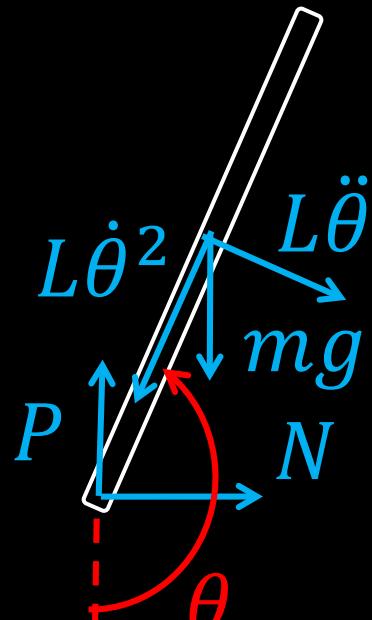


Inverted Pendulum on a Cart

- State Space

Free-body
diagrams

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$



Linearized about $\theta = \pi$

- (10) $F = u = (M + m)\ddot{x} + d\dot{x} - mL\ddot{\varphi}$
- (11) $L\ddot{\varphi} - g\varphi = \ddot{x}$

- Substitute (11) into (10)

- (12) $(M + m)(\ddot{\varphi}L - g\varphi) + d\dot{x} - mL\ddot{\varphi} = u$
- (13) $M\ddot{\varphi}L - (M + m)g\varphi + d\dot{x} = u$

$$\bullet \quad (14) \quad \ddot{\varphi} = \frac{M+m}{ML} g\varphi - \frac{d}{ML} \dot{x} + \frac{1}{ML} u$$

- Plug (14) back into (11)

$$\bullet \quad (15) \quad \ddot{x} = \frac{(M+m)g}{M} \varphi - \frac{d}{M} \dot{x} + \frac{1}{M} u - g\varphi$$

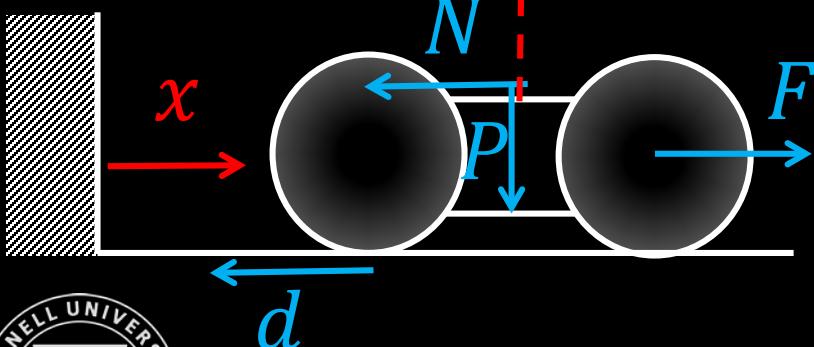
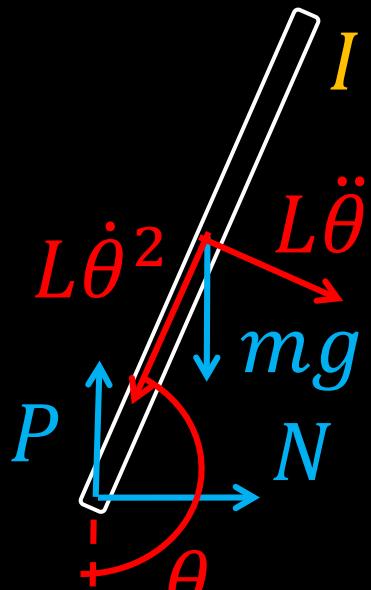
$$\bullet \quad (16) \quad \ddot{x} = \frac{m}{M} g\varphi - \frac{d}{M} \dot{x} + \frac{1}{M} u$$

Inverted Pendulum on a Cart

- State Space

Free-body diagrams

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$



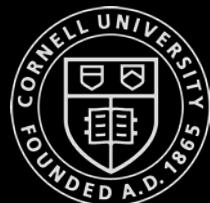
$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{M} & \frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{d}{ML} & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u$$

Linearized about $\theta = \pi$

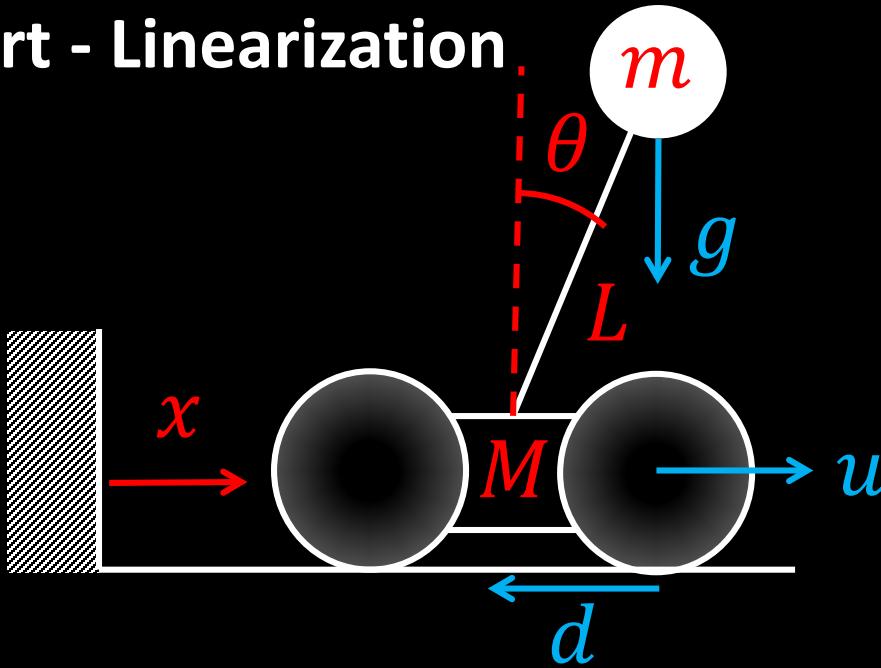
- (14) $\ddot{\varphi} = \frac{(M+m)g}{ML} \varphi - \frac{d}{ML} \dot{x} + \frac{1}{ML} u$

- (16) $\ddot{x} = \frac{m}{M} g \varphi - \frac{d}{M} \dot{x} + \frac{1}{M} u$

- $\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{M} & \frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{d}{ML} & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u$



Inverted Pendulum on a Cart - Linearization



Eq. of motion

↓
State space
model

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

→ Fixed points, \bar{x}

$$\begin{aligned} \theta &= 0, \pi \quad \text{down} \\ \dot{\theta} &= 0 \quad \text{up} \\ \dot{x} &= 0 \\ x &\text{ free variable} \end{aligned}$$

→ Jacobian

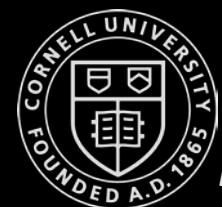
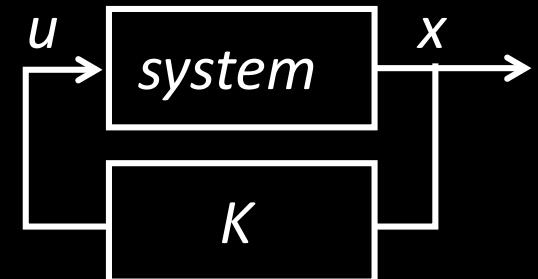
$$\frac{df}{dx} \Big|_{\bar{x}}$$

$$\dot{x} = Ax + Bu$$

Force acting on the
cart in the x direction

(A,B) Controllable?

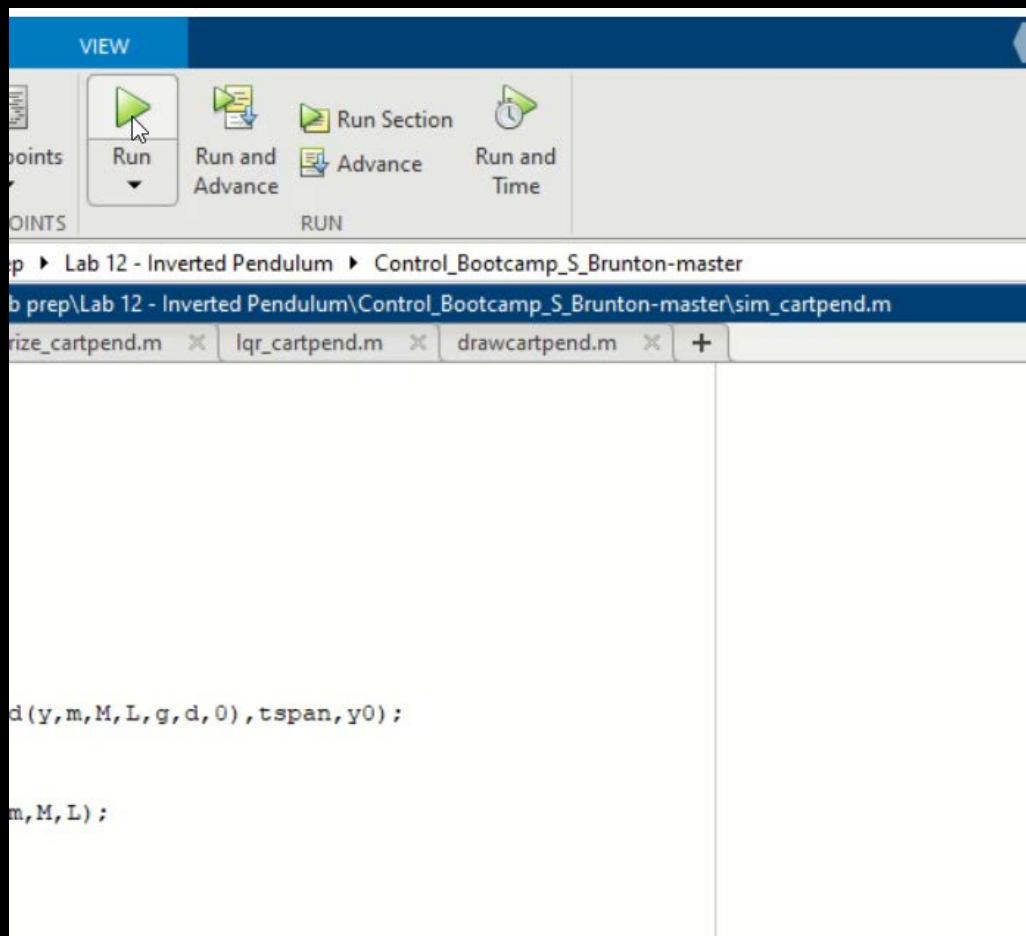
↓
Add linear control
 $\dot{x} = (A - BK)x$



Inverted Pendulum on a Cart - Linearization

Matlab example

- Non-linear model
- Linearized model
- Eigenvalues
- Stability
- Controllability

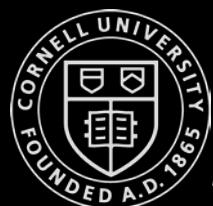


The screenshot shows a Matlab interface. The toolbar at the top has a 'Run' button highlighted with a cursor. The menu bar says 'VIEW'. The workspace window shows a file path: 'Lab 12 - Inverted Pendulum > Control_Bootcamp_S_Brunton-master > sim_cartpend.m'. Below the path, there are tabs for 'size_cartpend.m', 'lqr_cartpend.m', and 'drawcartpend.m'. The main workspace area contains the following code:

```
d(y,m,M,L,g,d,0),tspan,y0);  
  
m,M,L);
```

$$\gg eig(A)$$
$$\lambda_4 = 3.5069 \lambda_3 = -1.9$$

$$\gg rank(ctrb(A,B))$$
$$4$$

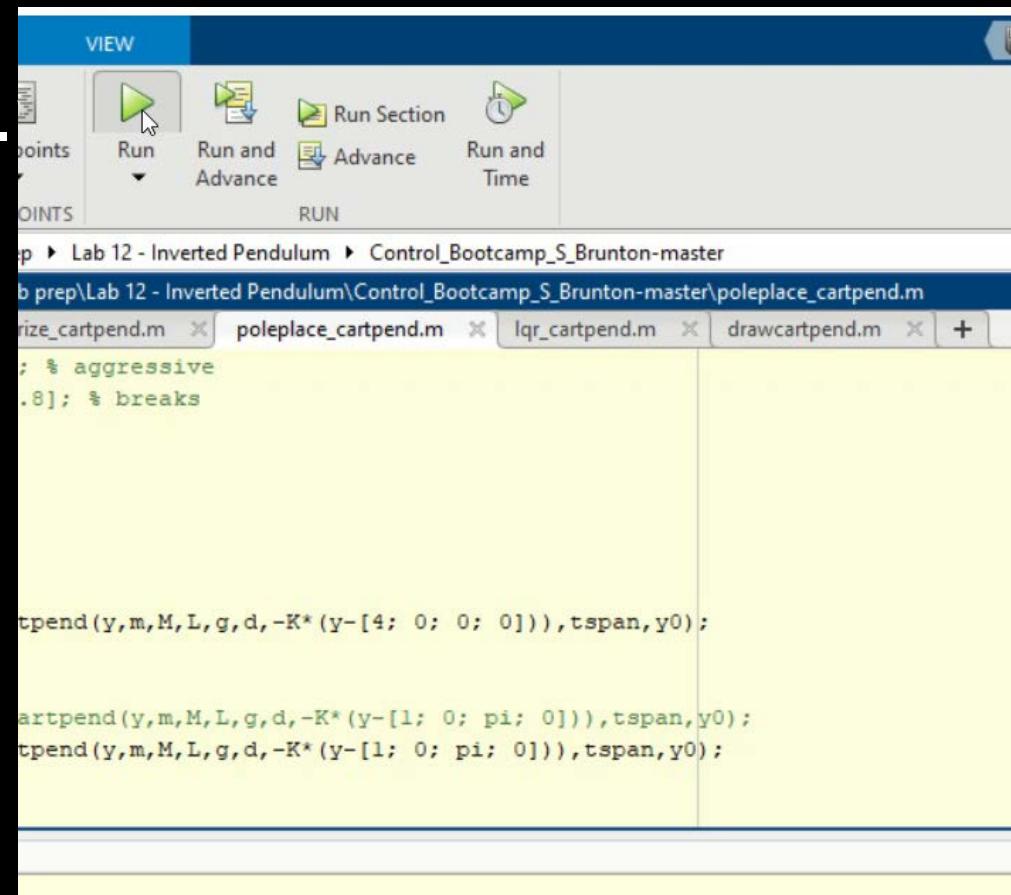


Inverted Pendulum on a Cart - Linearization

Matlab example

- Non-linear model
- Linearized model
- Eigenvalues
- Stability
- Controllability
- Add control

```
>> eigs = [-1.1;-1.2;-1.3;-1.4]
>> K = place(A,B,eigs)
K=[-0.0965 -1.3111  8.7254  2.2295]
>> eig(A-B.*K)
[-1.4; -1.3; -1.2; -1.
```

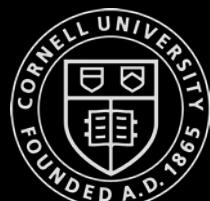


The screenshot shows a MATLAB interface. The toolbar at the top includes buttons for Run, Run and Advance, Run Section, and Run and Time. Below the toolbar is a menu bar with 'VIEW' selected. A file browser window is open, showing the path: 'Lab 12 - Inverted Pendulum > Control_Bootcamp_S_Brunton-master'. Inside this folder, several files are listed: 'lqr_cartpend.m' (selected), 'poleplace_cartpend.m', 'drawcartpend.m', and others. The code editor window displays MATLAB code related to the inverted pendulum. The visible code includes:

```
% aggressive
.8]; % breaks

tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);

artpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
tpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
```



Pole Placement

- In Python
 - https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.place_poles.html
 - $K = \text{scipy.signal.place_poles}(A, B, \text{poles})$
- Barely stable eigenvalues
 - Not enough control authority
- More negative eigenvalues
 - Faster dynamics
 - Less robust system
- Linear Quadratic Control (LQR)
 - “Sweet spot of eigenvalues”
 - Balances how fast you stabilize your state and how much control energy you spend to get there

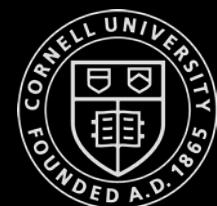
Linear Quadratic Control

$$\dot{x} = Ax + Bu, x \in \mathbb{R}^n$$

$$u = -Kx$$

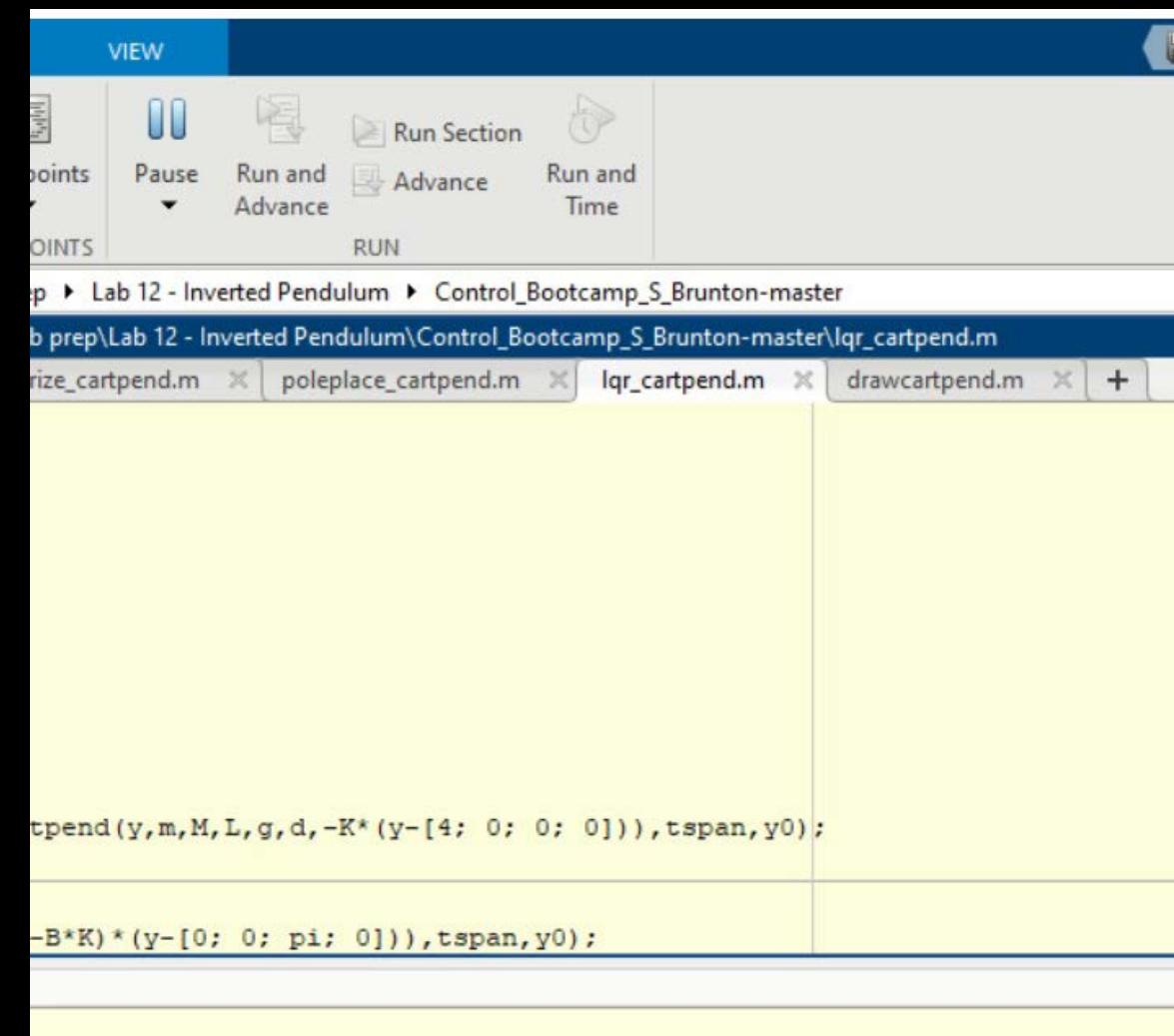
$$\dot{x} = (A - BK)x$$

- >> K = place(A,B,eigs)
- Where are the best eigs??
 - Linear Quadratic Regulator (LQR)
 - >> K = lqr(A,B,Q,R)
 - $\int_0^\infty (x^T Q x + u^T R u) dt$
 - $Q = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 10 \end{bmatrix}, R = 0.001$
 - Ricotta equation
 - Computationally expensive, $O(n^3)$



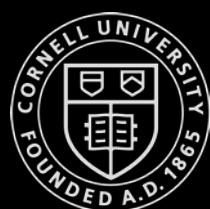
Matlab Example

- $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}, R = 0.001$
- $K = lqr(A, B, Q, R);$
- $\gg [T, D] = eigs(A - B.*K)$
 - $\lambda_1 = -788.29 + 0.00i$
 - $\lambda_2 = -0.70 + 0.83i$
 - $\lambda_3 = -0.70 - 0.83i$
 - $\lambda_4 = -0.83 + 0.00i$
- $T(:,1)$
 - $= [0.0008, -0.6387, 0.0010, -0.7695]^T$



The screenshot shows a Matlab interface. The toolbar at the top includes buttons for Pause, Run and Advance, Run Section, Advance, and Run and Time. Below the toolbar is a navigation bar showing the path: 'Lab 12 - Inverted Pendulum > Control_Bootcamp_S_Brunton-master'. The main area is a code editor with several tabs open: 'lqr_cartpend.m' (which is currently active), 'drawcartpend.m', 'poleplace_cartpend.m', and 'size_cartpend.m'. The code in the active tab is as follows:

```
tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);  
-B*K)*(y-[0; 0; pi; 0])),tspan,y0);
```



Matlab Example

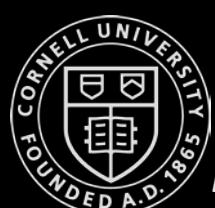
- $Q = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 10 \\ & & 100 \end{bmatrix}, R = 0.001$
- $K = \text{lqr}(A, B, Q, R);$
- $\gg [T, D] = \text{eigs}(A - B.*K)$
 - $\lambda_1 = -788.29 + 0.00i$
 - $\lambda_2 = -0.70 + 0.83i$
 - $\lambda_3 = -0.70 - 0.83i$
 - $\lambda_4 = -0.83 + 0.00i$
- $T(:,1)$
 - $= [0.0008, -0.6387, 0.0010, -0.7695]^T$

$$\begin{aligned}\lambda_1 &= -25.6851 + 0.0000i \\ \lambda_2 &= -1.0855 + 0.8921i \\ \lambda_3 &= -1.0855 - 0.8921i \\ \lambda_4 &= -0.4811 + 0.0000i\end{aligned}$$

The screenshot shows a Matlab interface with the following details:

- Toolbar:** VIEW, Pause, Run and Advance, Advance, Run and Time.
- Path:** /Lab 12 - Inverted Pendulum /Control_Bootcamp_S_Brunton-master
- Open Files:** lqr_cartpend.m (active), drawcartpend.m, poleplace_cartpend.m, rize_cartpend.m.
- Code Preview:** The code block contains three lines of M-code related to an inverted pendulum simulation.

```
tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);  
-B*K)*(y-[0; 0; pi; 0])),tspan,y0);  
cartpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
```



Linear Quadratic Control

- `>> K = place(A,B,eigs)`
- Where are the best eigs??

$$\dot{x} = Ax + Bu, x \in \mathbb{R}^n$$

$$u = -Kx$$

$$\dot{x} = (A - BK)x$$

- Linear Quadratic Regulator (LQR)

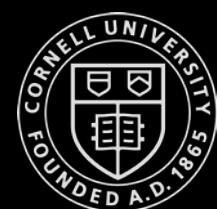
- `>> K = lqr(A,B,Q,R)`

- $\int_0^\infty (x^T Q x + u^T R u) dt$

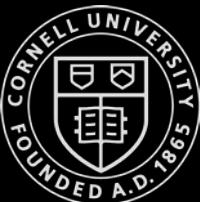
- $Q = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 10 \end{bmatrix}, R = 0.001$

- Riccati equation
 - Computationally expensive, $O(n^3)$

- *The linear controller works!*
 - *(in simulation)*
- *Issues in Practice?*
 - *Imperfect models*
 - *Nonlinear parts*
 - *Deadband, saturation, etc.*
 - *We don't have full state feedback*



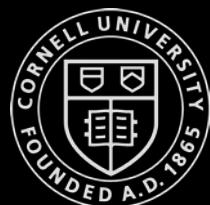
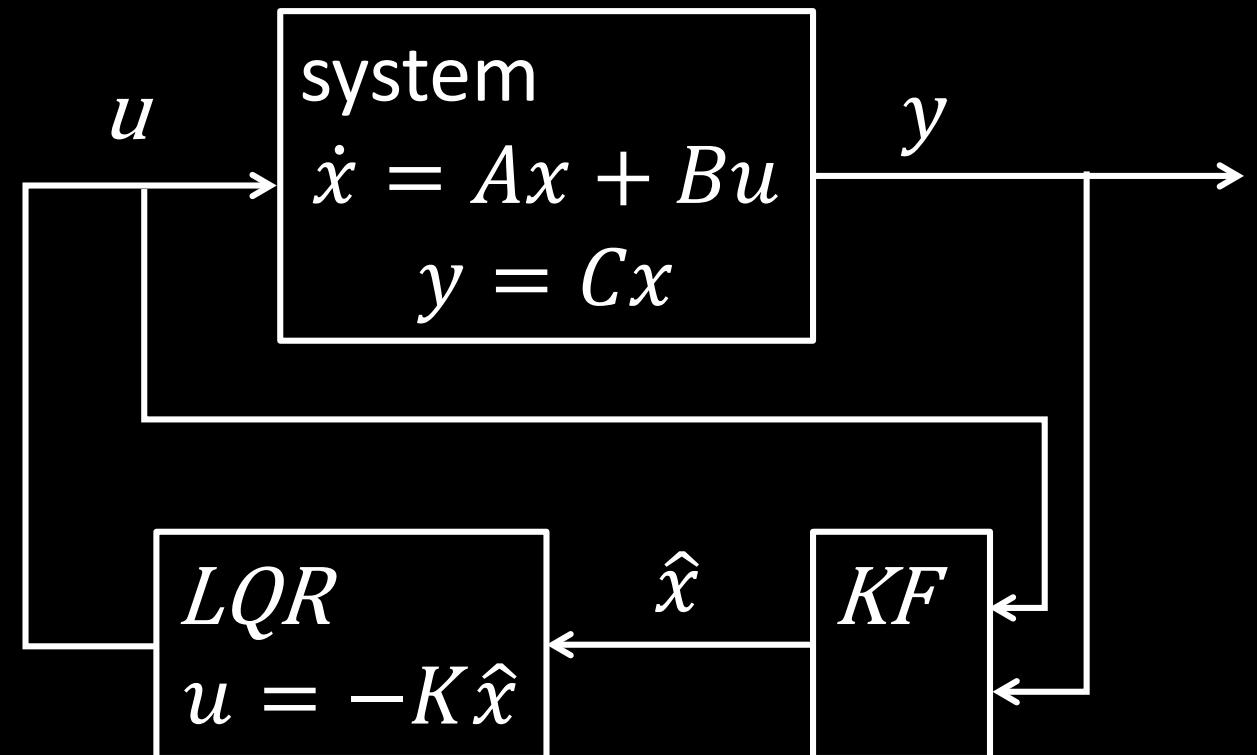
Full State Feedback



Full State Feedback

- Controllability
 - Can we steer the system anywhere given some control input u ?
- Observability
 - Can we estimate any state x , from a time series of your measurements $y(t)$?

$$\begin{aligned}\dot{x} &= Ax + Bu, x \in \mathbb{R}^n \\ u &= -Kx \\ \dot{x} &= (A - BK)x\end{aligned}$$



Any last questions on Lab 10?

