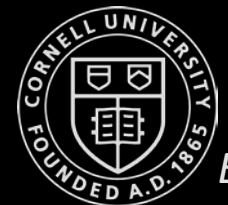


ECE 4960

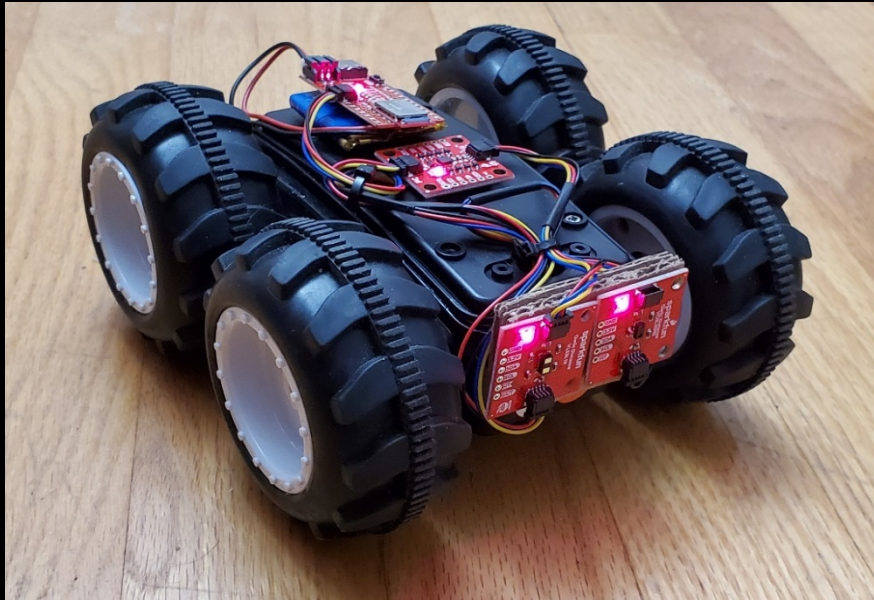
Prof. Kirstin Hagelskjær Petersen
kirstin@cornell.edu

Fast Robots



Robot Configurations

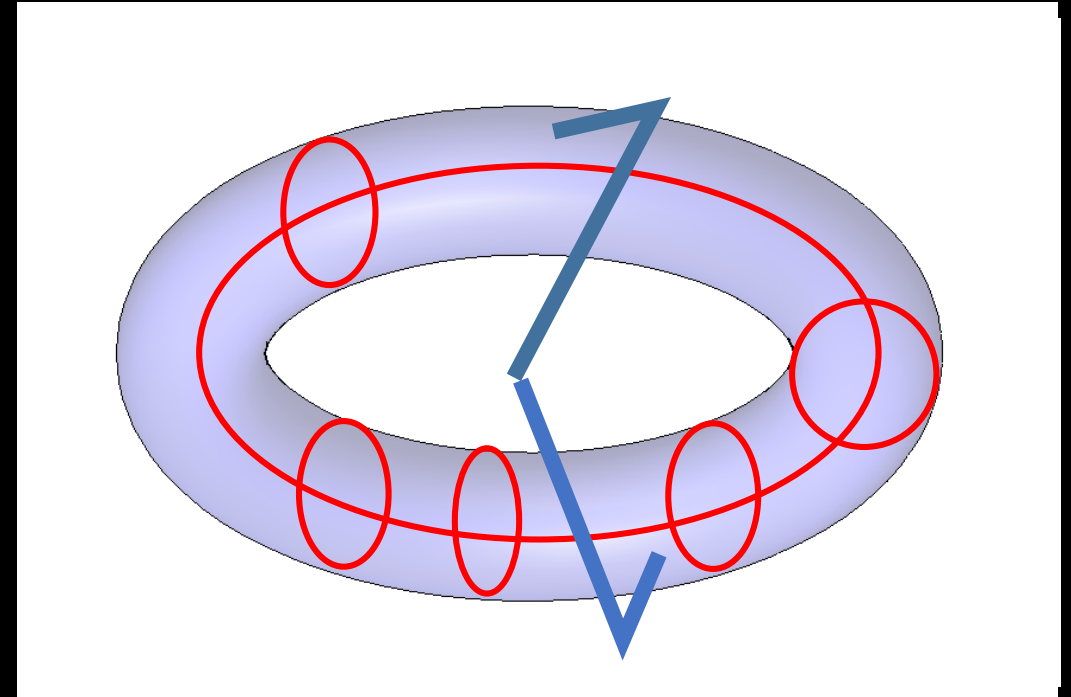
- Objective: Understand coordinate transformations as used with robots
 - “Rigid-body kinematics”
- Robot configuration specifies all points on the robot
- The robot C-space is the space of all configurations
- The DOF is the dimension of the C-space



What is the DOF of these?

Configuration, Configuration space, Degrees of Freedom

- 2 DOF robot arm
 - C-space: 2 angles
 - J-space: Surface of a torus



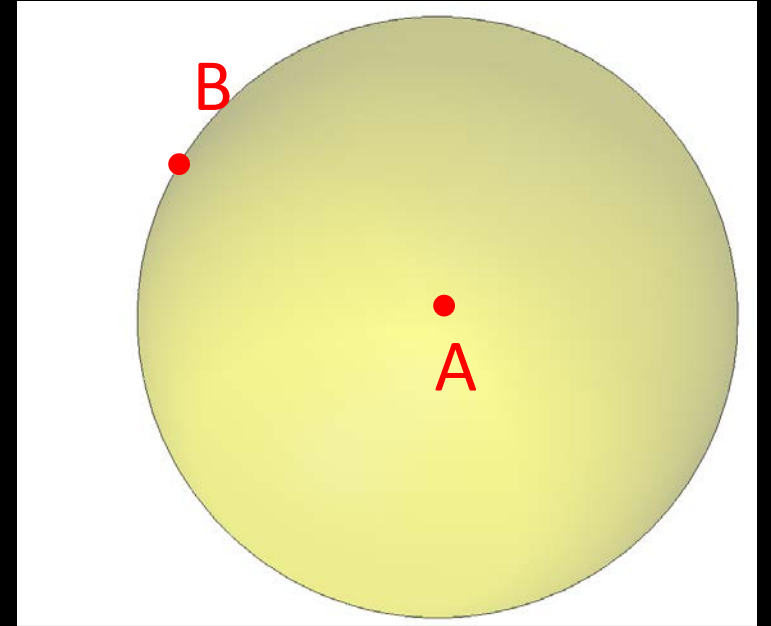
Robot Configurations

- Point A: $\{x, y, z\}$



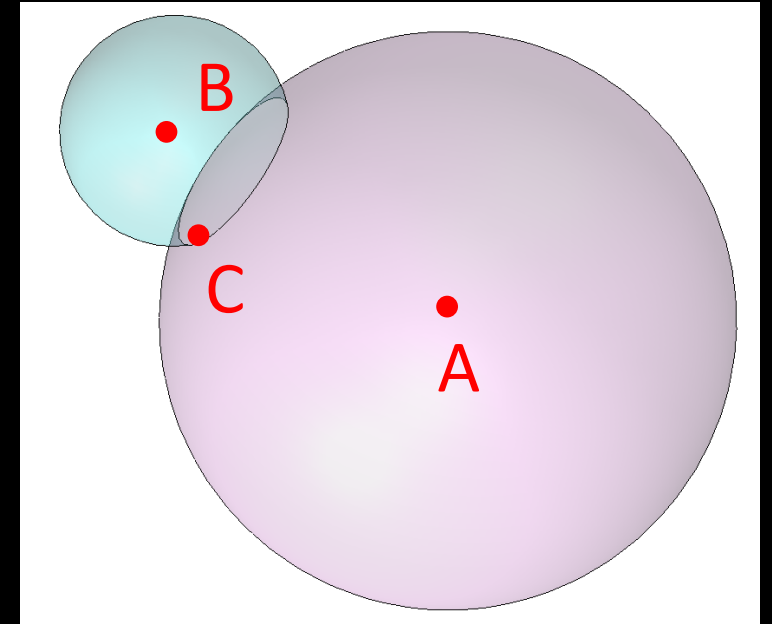
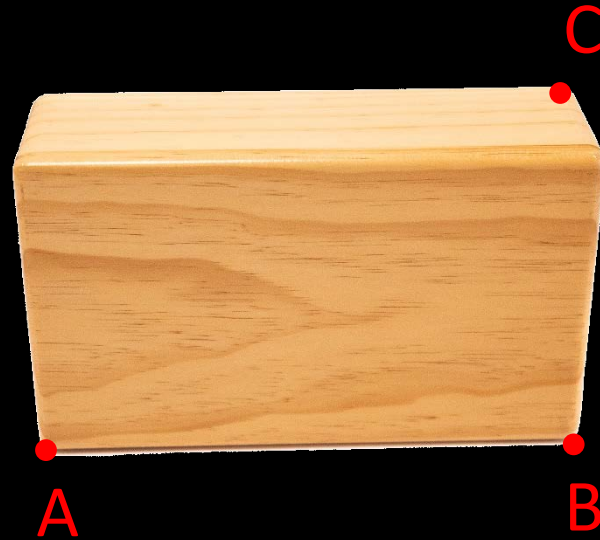
Robot Configurations

- Point A: $\{x, y, z\}$
- Point B: $\{\theta, \phi\}$



Robot Configurations

- Point A: $\{x, y, z\}$
- Point B: $\{\theta, \phi\}$
- Point C: $\{\psi\}$
- A rigid body in 3D has 6 DOF
- A rigid body in 2D has 3 DOF
- A rigid body in 4D has 10 DOF



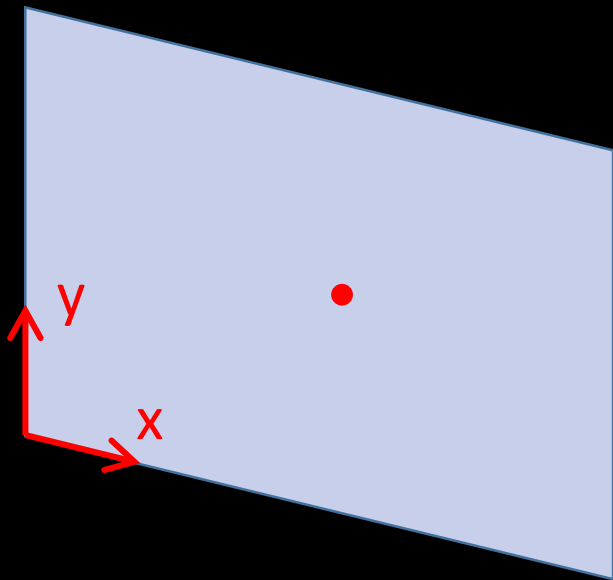
| Point | Coords | Ind. constraints | Real freedoms |
|-------|--------|------------------|---------------|
| A | 3 | 0 | 3 |
| B | 3 | 1 | 2 |
| C | 3 | 2 | 1 |
| D | 3 | 3 | 0 |
| Total | | | 6 |

$DOF = \Sigma(\text{freedoms of points}) - \text{no. of independent constraints}$

$DOF = \Sigma(\text{freedoms of bodies}) - \text{no. of independent constraints}$

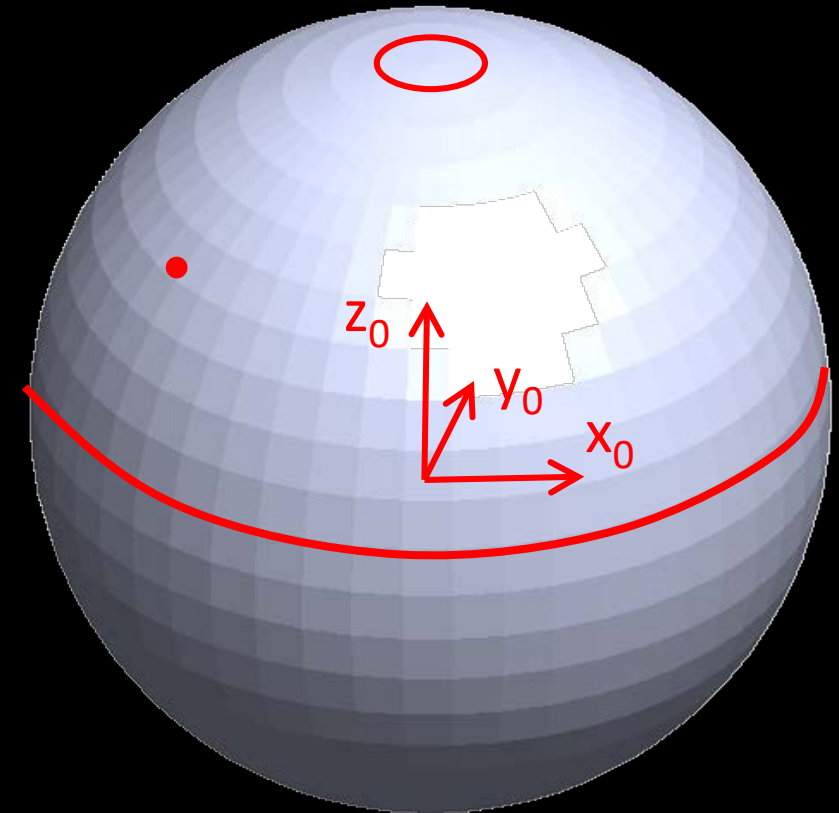
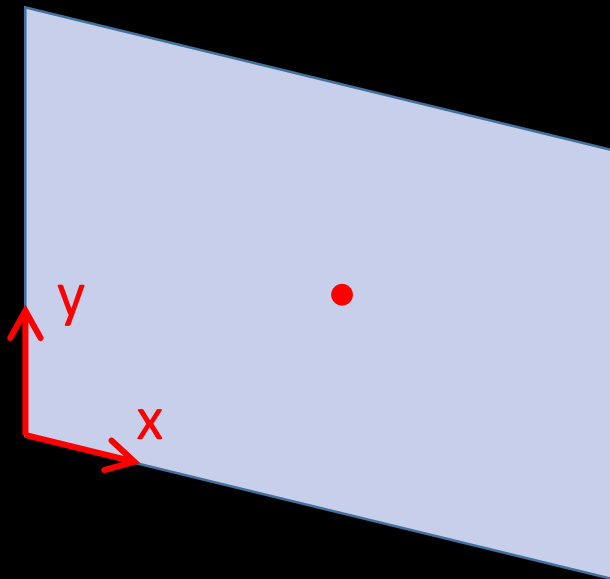
Topology Representation

- Point on a plane
 - Origin and 2 orthogonal coordinate axis



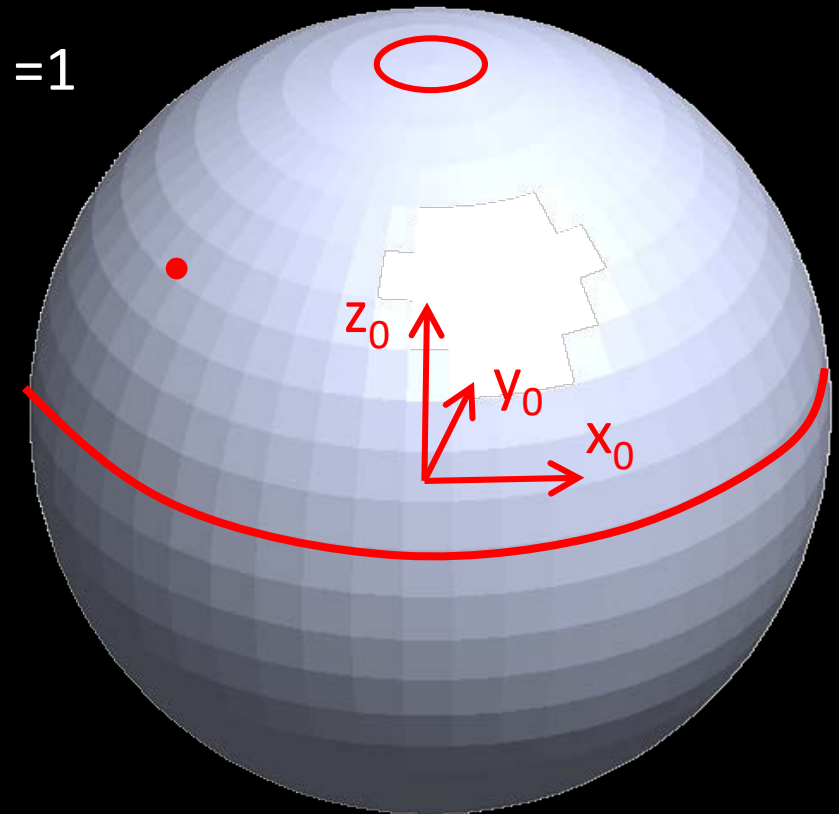
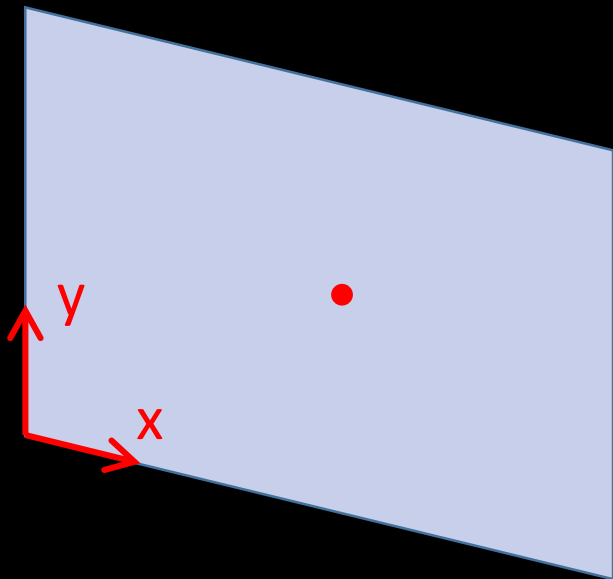
Topology Representation

- Point on a plane
 - Origin and 2 orthogonal coordinate axis
- Points on the surface of a sphere
 - “Explicit representation”: Latitude and longitude



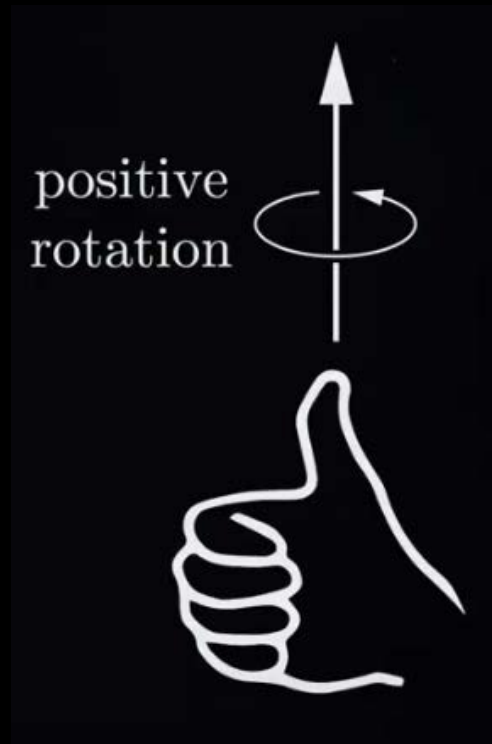
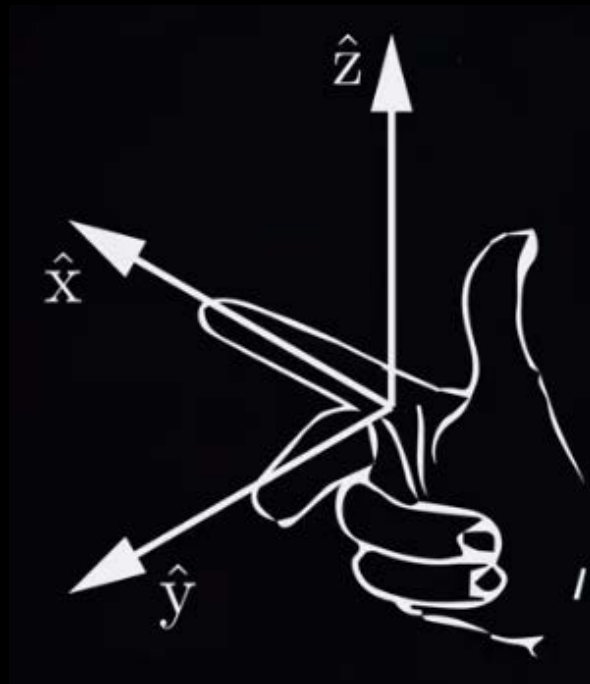
Topology Representation

- Point on a plane
 - Origin and 2 orthogonal coordinate axis
- Points on the surface of a sphere
 - “Explicit representation”: Latitude and longitude
 - “Implicit representation”: $\{X, Y, Z\}$, such that $x^2+y^2+z^2 = 1$
 - Slightly more complex, but singularity free!
 - 3D \rightarrow Rotation matrix

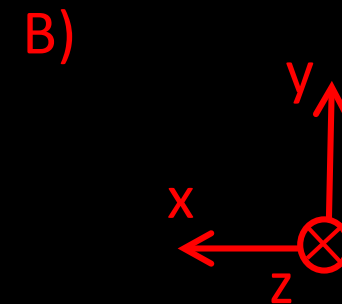
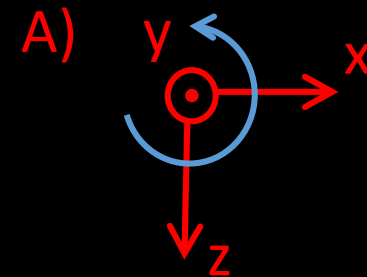


Coordinate Frames

- Reference frames (origin and $\{x, y, z\}$ -coordinates)
- Right hand frames and rotations

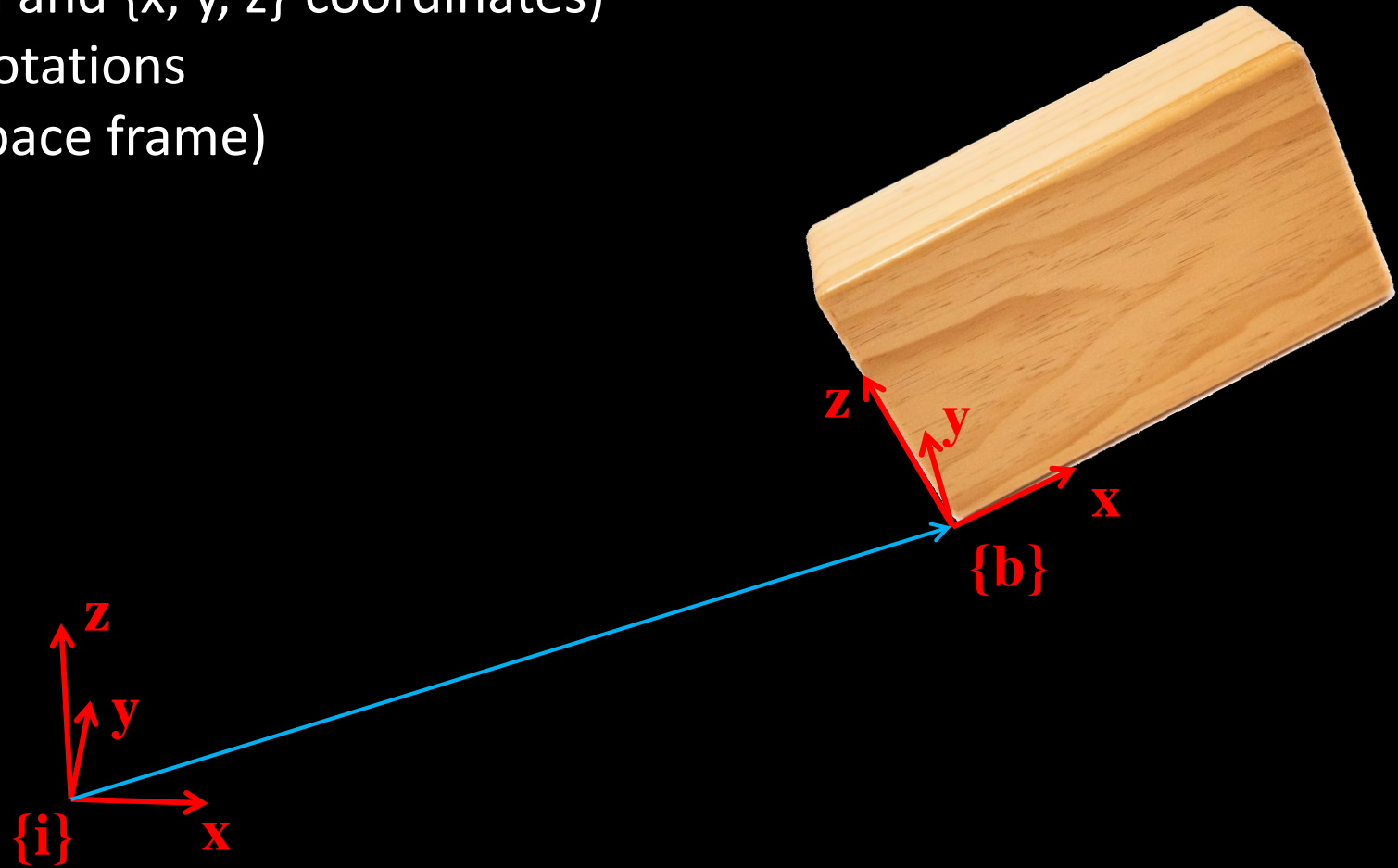


Are these right-handed or left-handed?



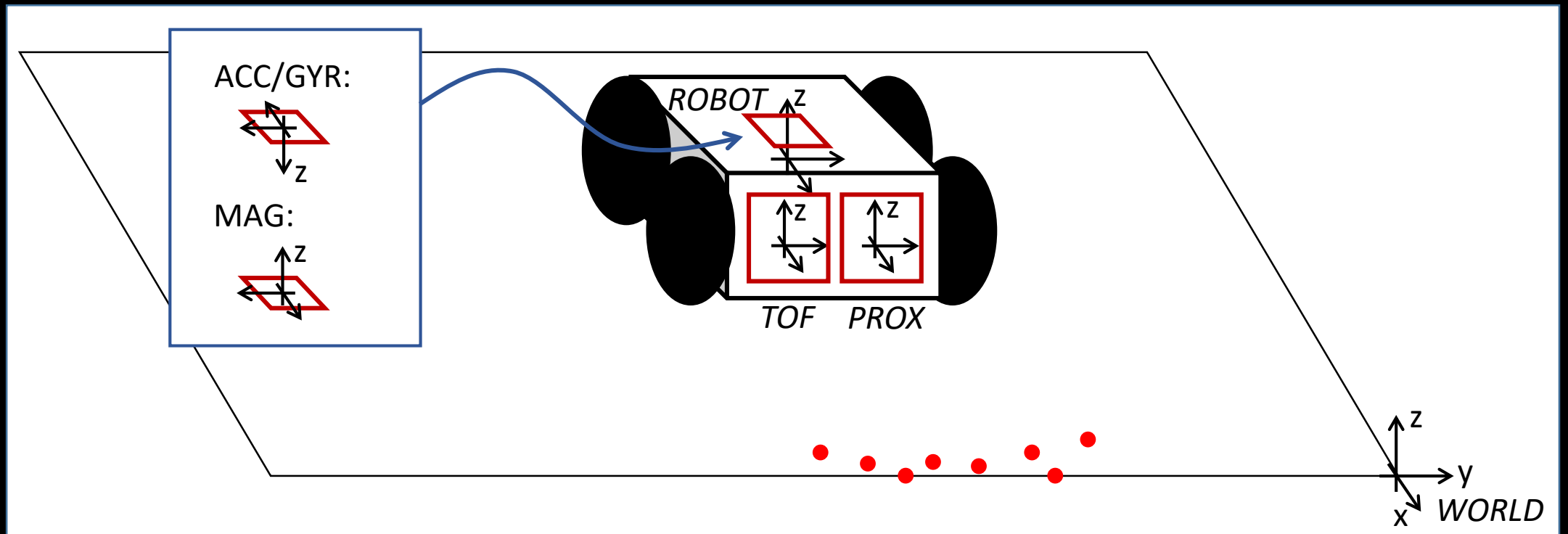
Coordinate Frames

- Reference frames (origin and $\{x, y, z\}$ -coordinates)
- Right hand frames and rotations
- Inertial frame (/world/space frame)
- Body frame



Coordinate Frames

- Reference frames (origin and $\{x, y, z\}$ -coordinates)
- Right hand frames and rotations
- Inertial frame (/world/space frame)
- Body frame

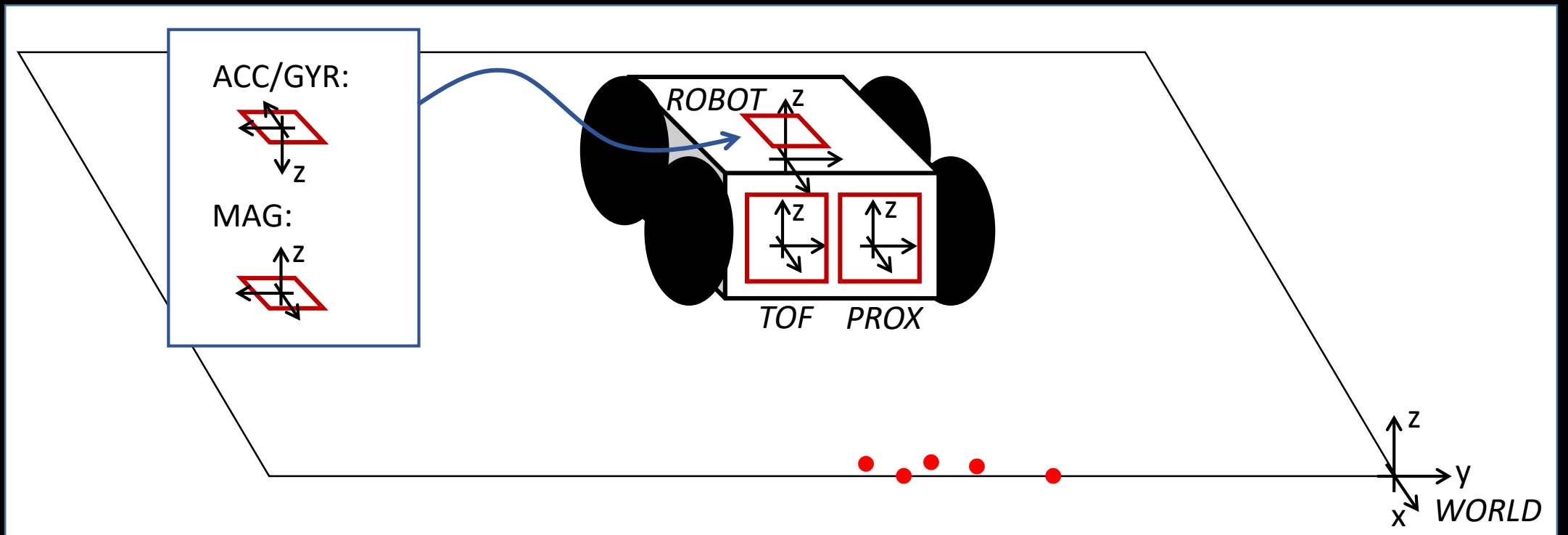


Homogeneous Transformation Matrix

rotation

translation

$$T = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & x \\ r_{21} & r_{22} & r_{23} & y \\ r_{31} & r_{32} & r_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



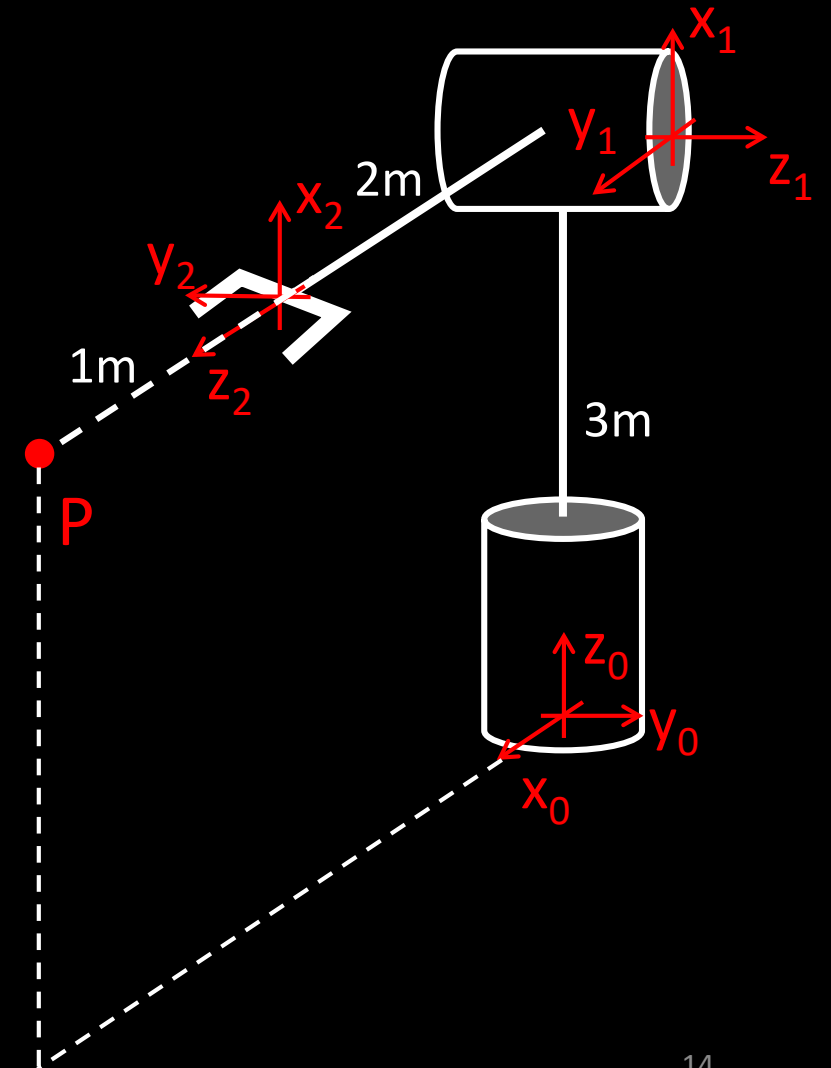
Homogeneous Transformation Matrix

- What is the location of the point P in reference frame 2?

$$P^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

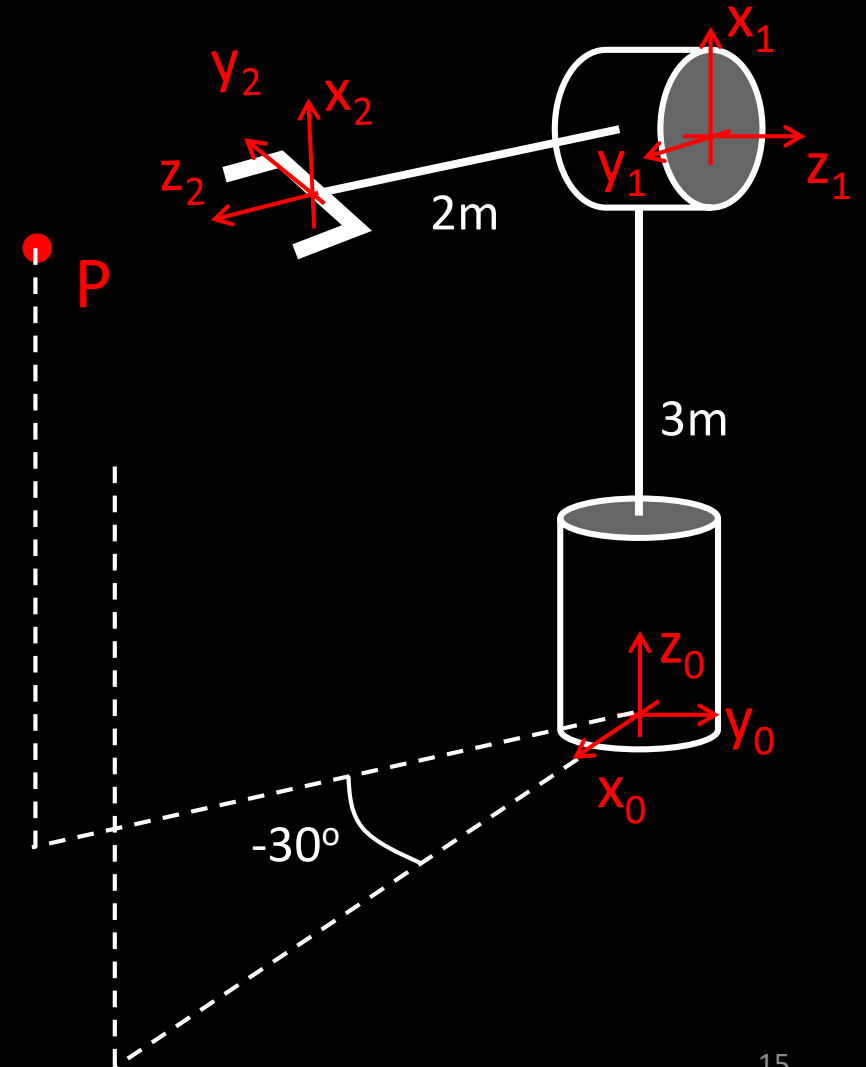
- What is the location of point P in reference frame 0?

$$P^0 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$



Homogeneous Transformation Matrix

- What is the location of the point P in reference frame 2?
- What is the location of point P in reference frame 0?

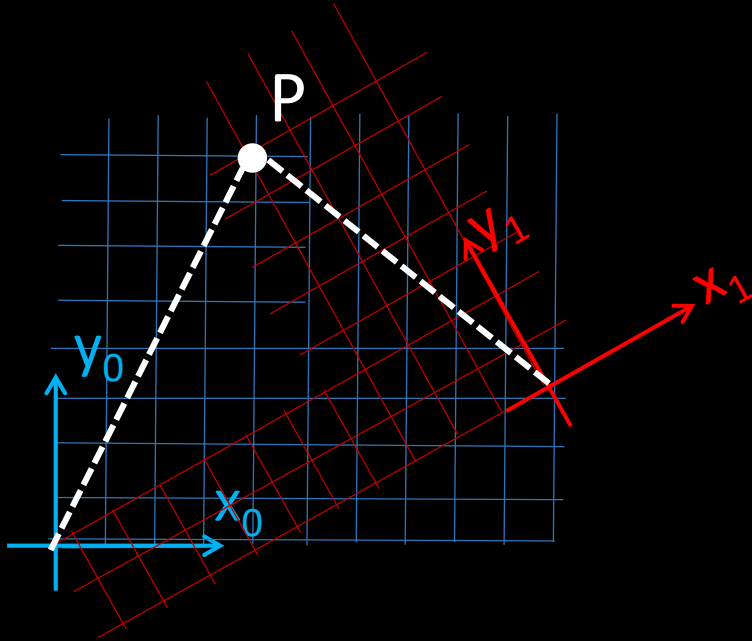


Homogeneous Transformation Matrix

- The change in position and orientation between frames is described using transformation matrices

$$P^0 = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \quad P^1 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$O_1^0 = \begin{bmatrix} 10 \\ 3.3 \end{bmatrix} \quad O_0^1 = \begin{bmatrix} -10.3 \\ 2 \end{bmatrix}$$



How do we express P^1 if we know P^0 and the relative location of O_0^1 ?

$$P^1 \neq P^0 + O_0^1$$

Homogeneous Transformation Matrix

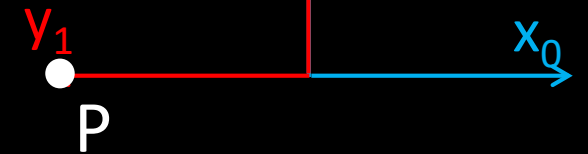
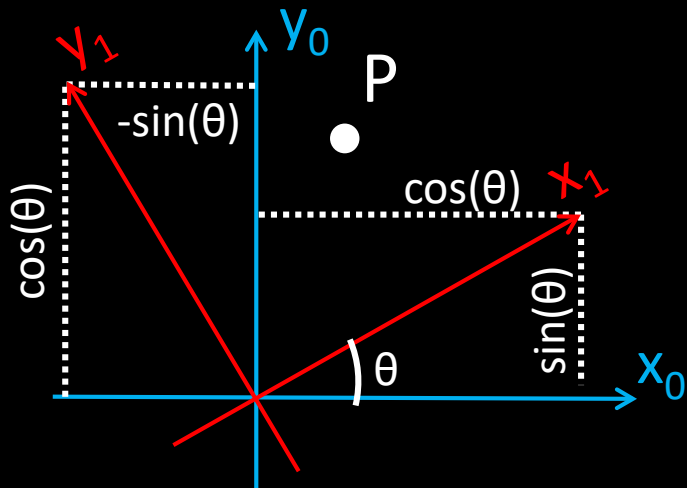
- We need both translation and rotation!

$$R_1^0 = [x_1^0 \quad y_1^0] \quad x_1^0 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad y_1^0 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \quad R_1^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$P^0 = R_1^0 P^1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} P^1$$

e.g. if $P^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\theta = 90^\circ$:

$$P^0 = R_1^0 P^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



Homogeneous Transformation Matrix

- We need both translation and rotation!

$$R_1^0 = [x_1^0 \quad y_1^0] \quad x_1^0 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad y_1^0 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \quad R_1^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



$$R_1^0 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$

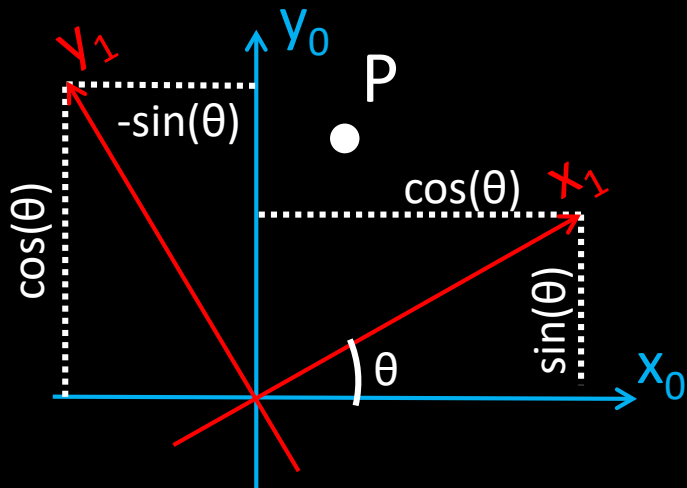
...Properties of rotation matrices:

$$R_0^1 = (R_1^0)^T = (R_1^0)^{-1} \quad \text{and} \quad \det(R) = 1$$

...really?

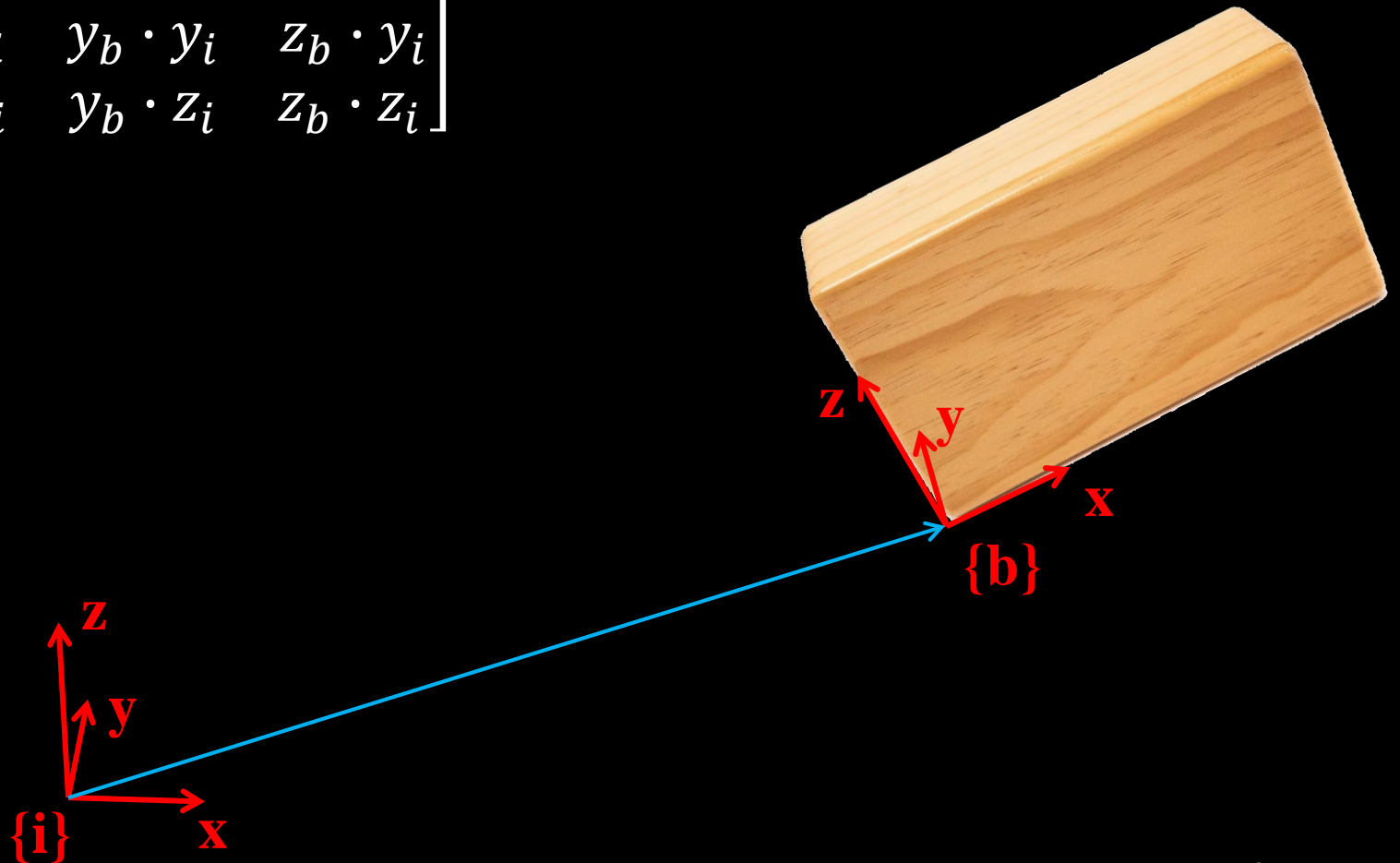
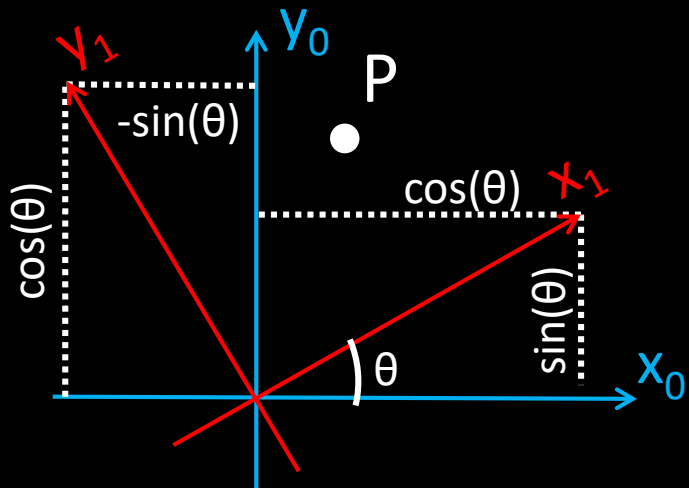
$$R_1^0 = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^T$$



Rotation Matrix in 3D

$$R_b^i = \begin{bmatrix} x_b \cdot x_i & y_b \cdot x_i & z_b \cdot x_i \\ x_b \cdot y_i & y_b \cdot y_i & z_b \cdot y_i \\ x_b \cdot z_i & y_b \cdot z_i & z_b \cdot z_i \end{bmatrix}$$

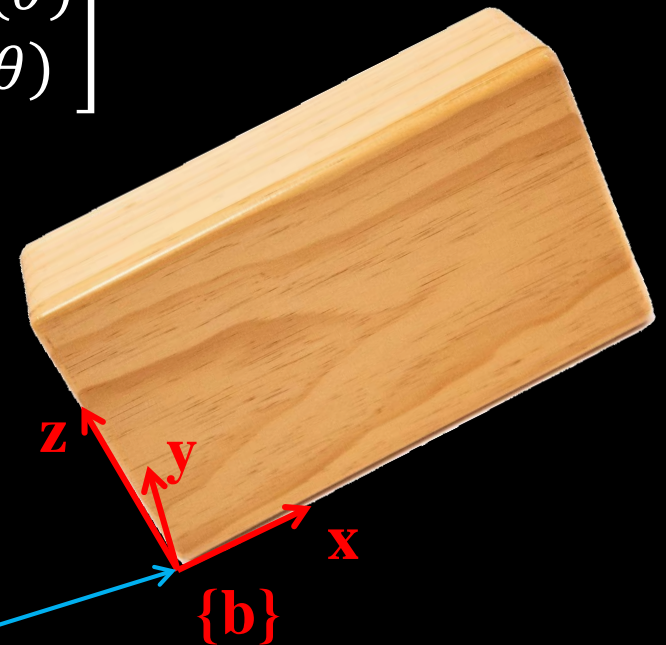
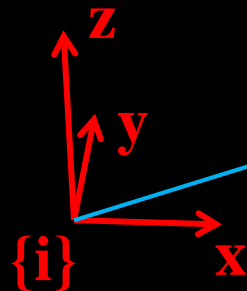
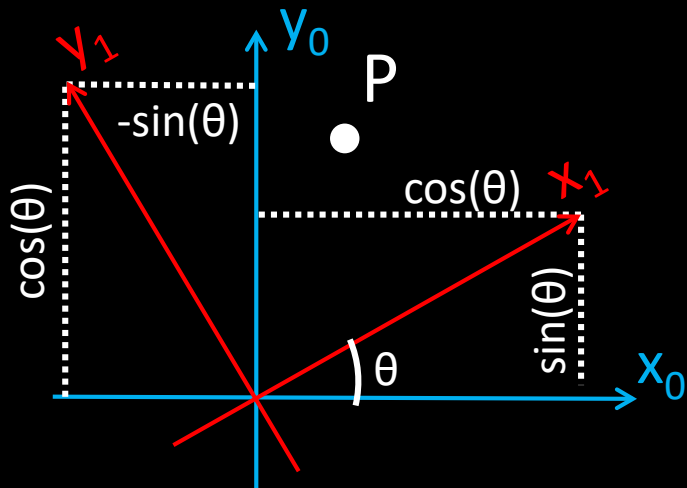


Rotation Matrix in 3D

- Find the rotation matrix $R_{z,\theta}$ for a rotation θ about z

$$R_{z,\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

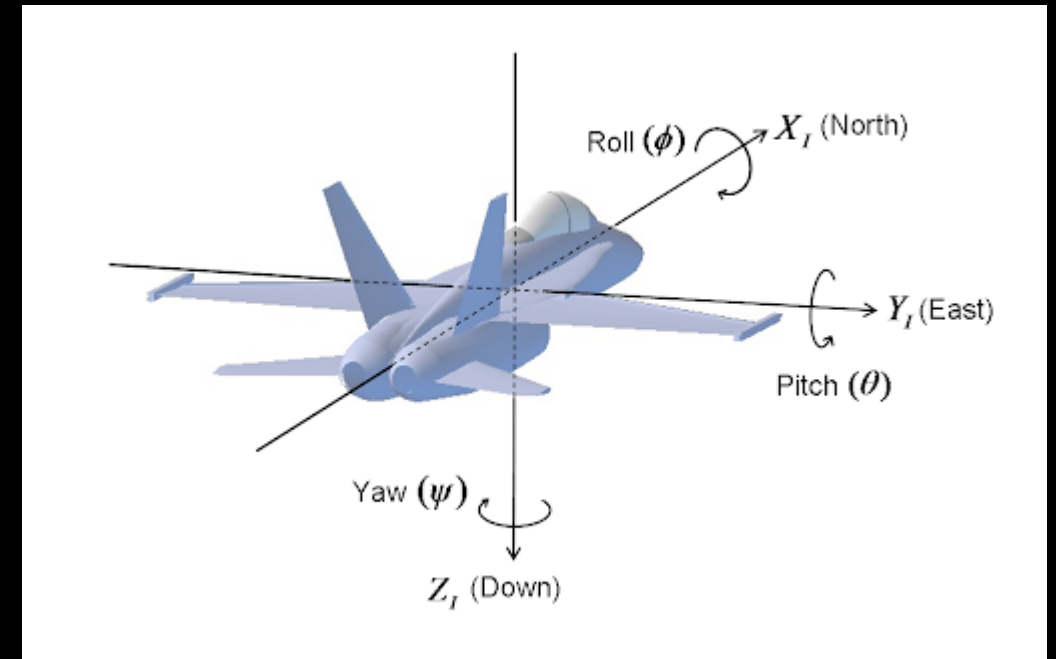
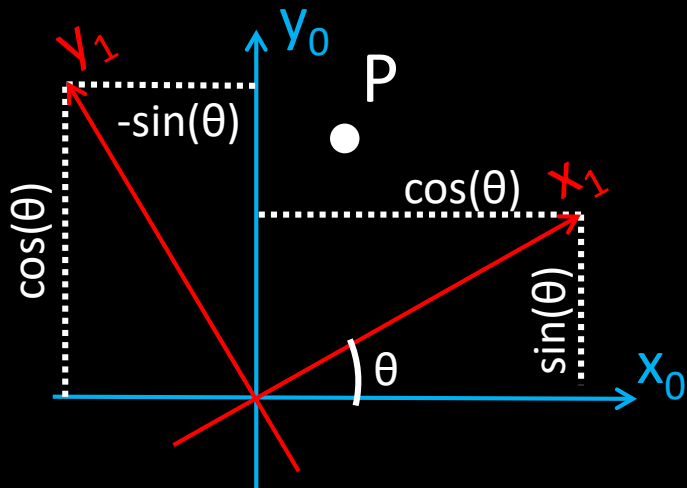


Rotation Matrix in 3D

- Find the rotation matrix $R_{z,\psi}$ for a rotation ψ about Z

$$R_{z,\psi} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

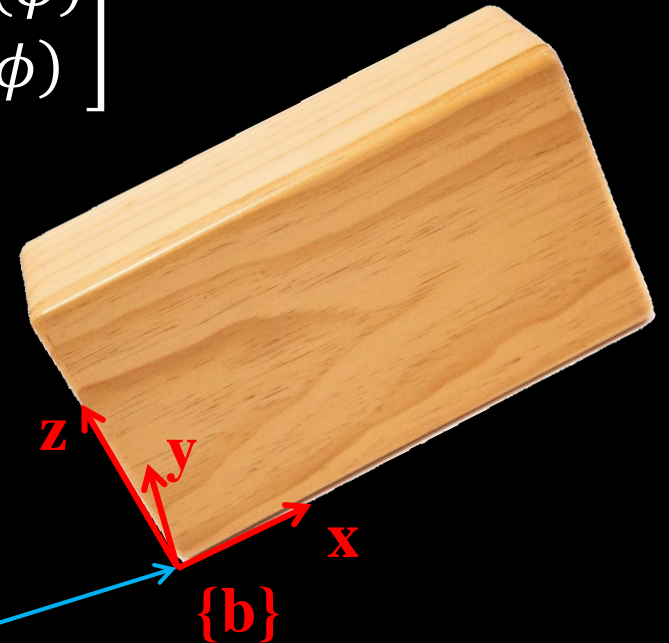
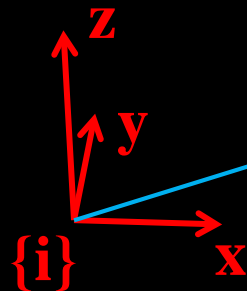
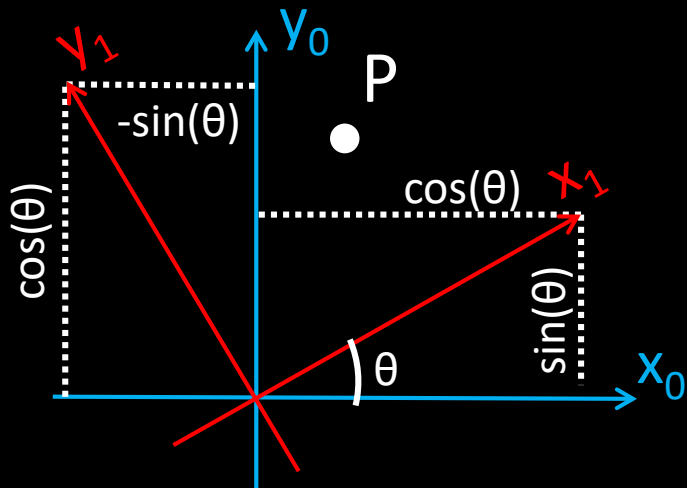


Rotation Matrix in 3D

- Find the rotation matrix $R_{z,\psi}$ for a rotation ψ about Z

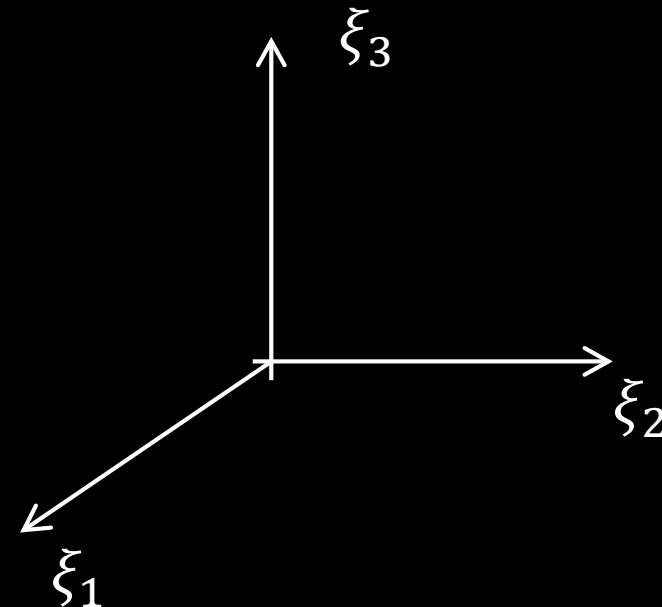
$$R_{z,\psi} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$



Euler

- “Any rotation can be described by three successive rotations about linearly independent axis.”
 - **Proper Euler angles**
 - $z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y$
 - **Tait–Bryan angles**
 - $x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z$
 - Most commonly $z-y-z$ or $x-y-z$

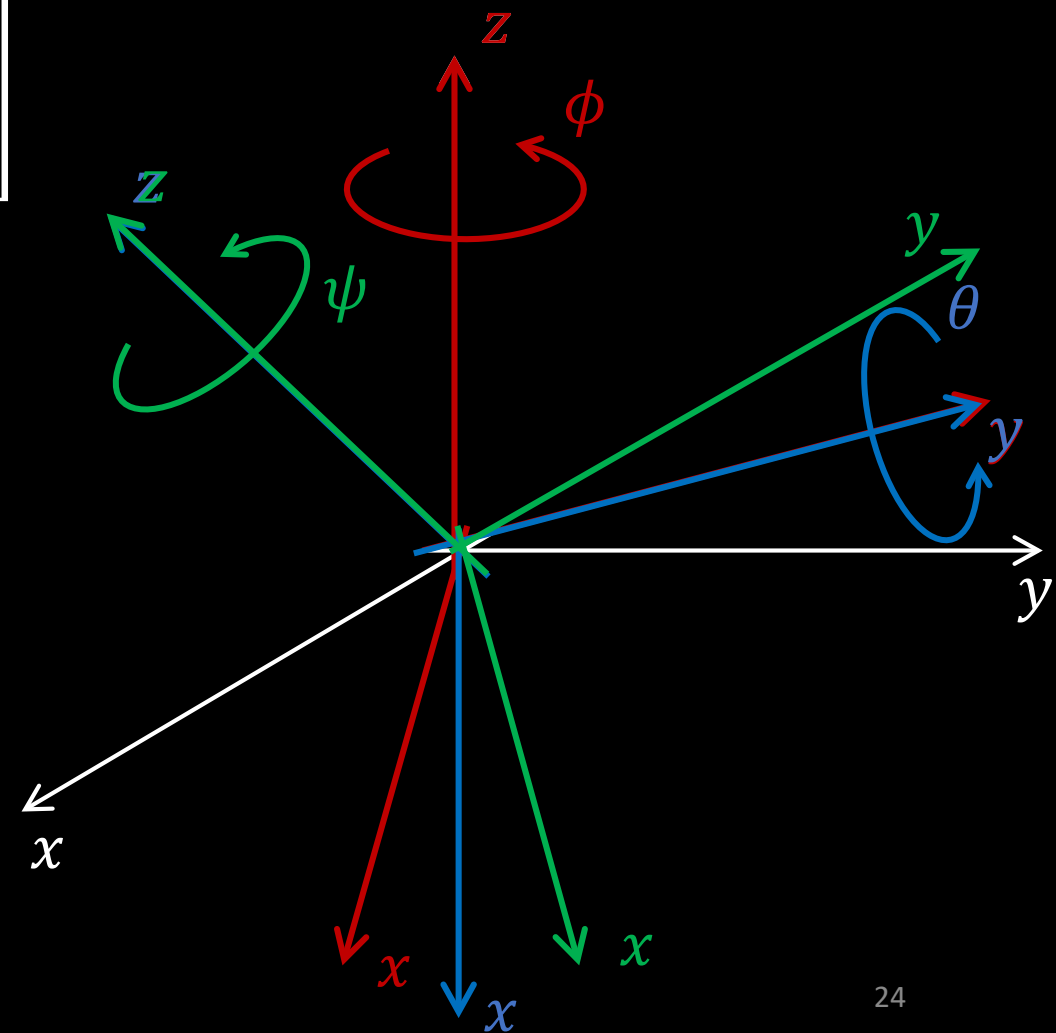


Rotation Matrix using ZYZ

$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\phi s_\psi & s_\phi s_\psi & c_\theta \end{bmatrix}$$



Rotation Matrix using Roll-Pitch-Yaw (X-Y-Z)

$$R_{XYZ} = R_{x,\phi} R_{y,\theta} R_{z,\psi}$$

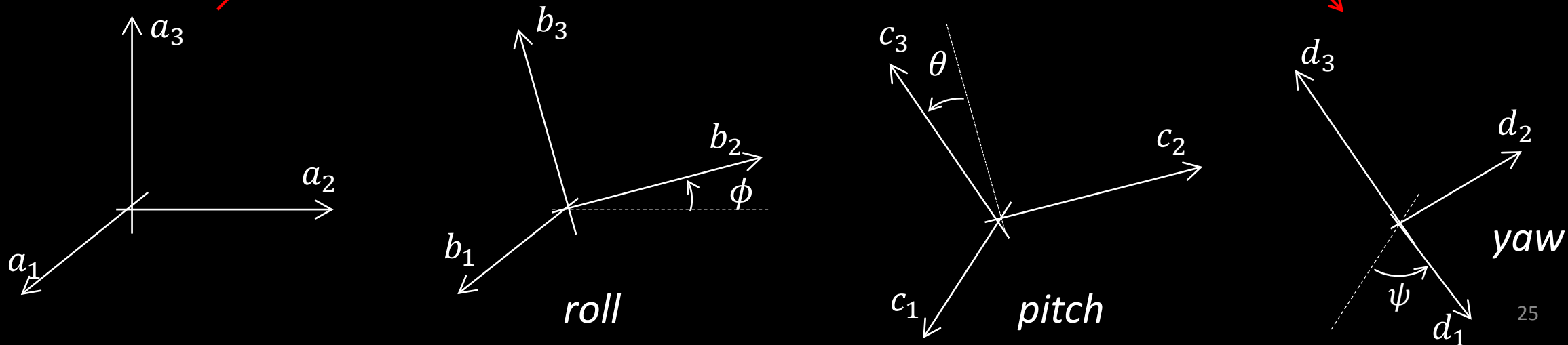
$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\phi s_\psi + s_\phi s_\theta c_\psi & c_\phi c_\psi - s_\phi s_\theta s_\psi & -s_\phi c_\theta \\ s_\phi s_\psi - c_\phi s_\theta c_\psi & s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\phi c_\theta \end{bmatrix}$$

$$R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_{z,\psi} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_D^A = R_B^A R_C^B R_D^C$$



Rotation Matrix using Roll-Pitch-Yaw (X-Y-Z)

$$R_{XYZ} = R_{x,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\phi s_\psi + s_\phi s_\theta c_\psi & c_\phi c_\psi - s_\phi s_\theta s_\psi & -s_\phi c_\theta \\ s_\phi s_\psi - c_\phi s_\theta c_\psi & s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\phi c_\theta \end{bmatrix}$$

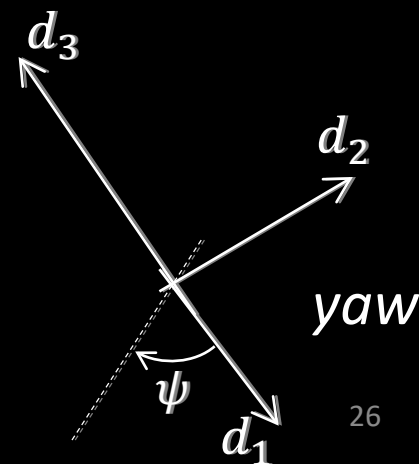
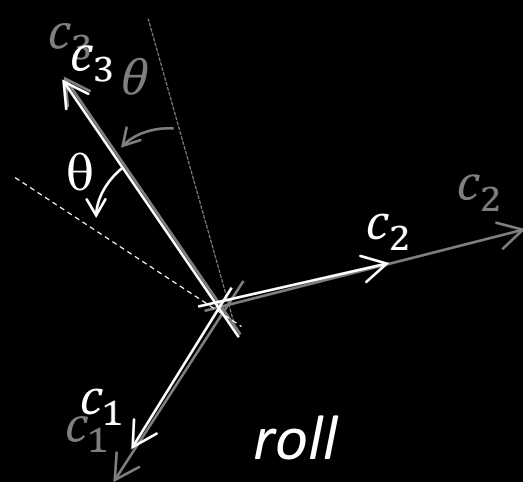
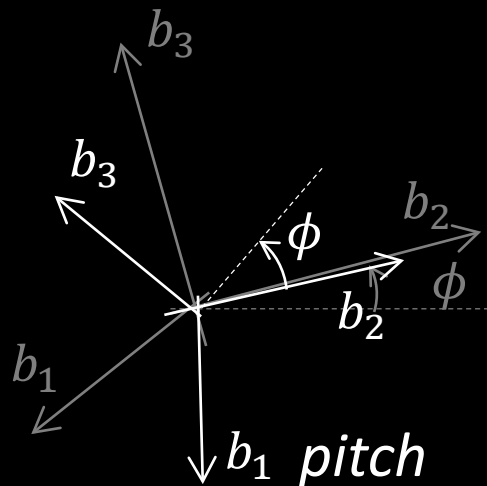
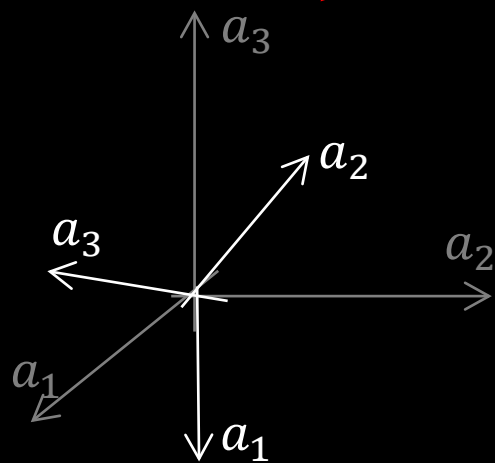
$$R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_{z,\psi} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Does the order matter? YES!

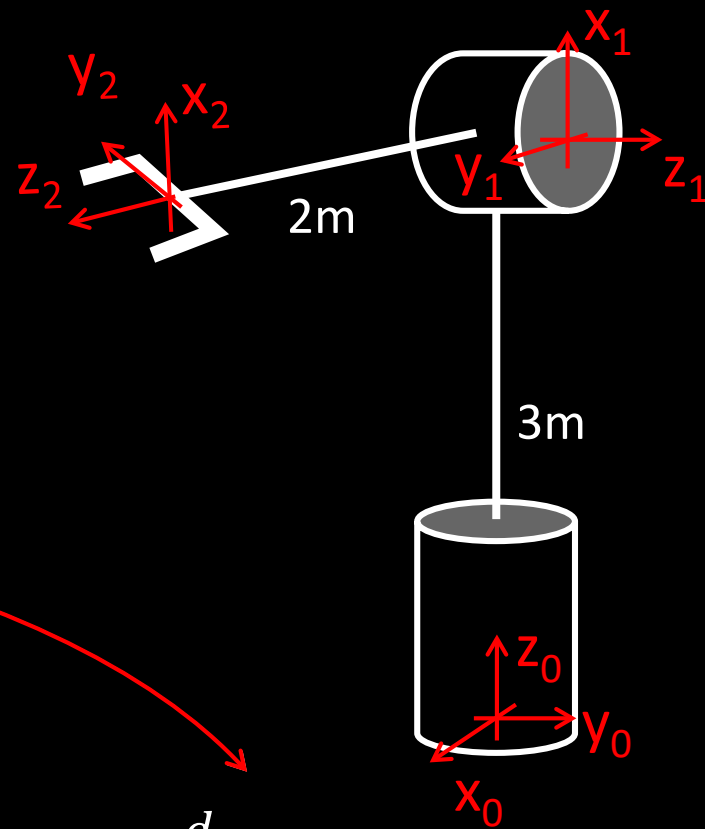
~~$R_D^A = R_D^C R_C^B R_B^A ?$~~



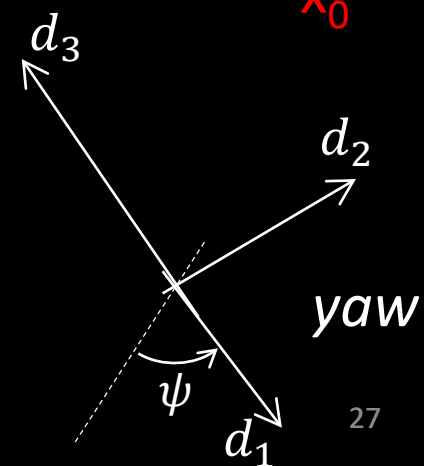
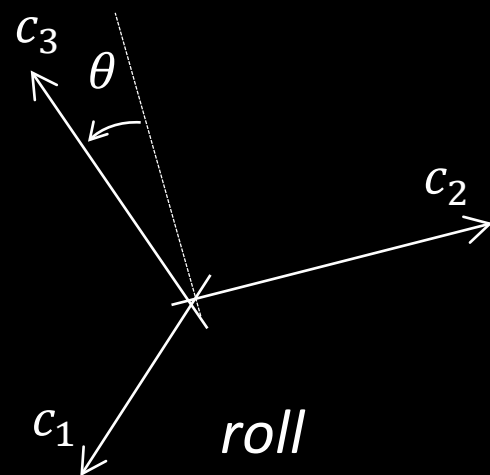
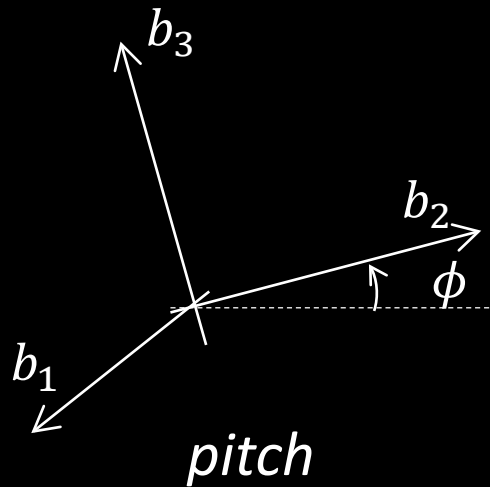
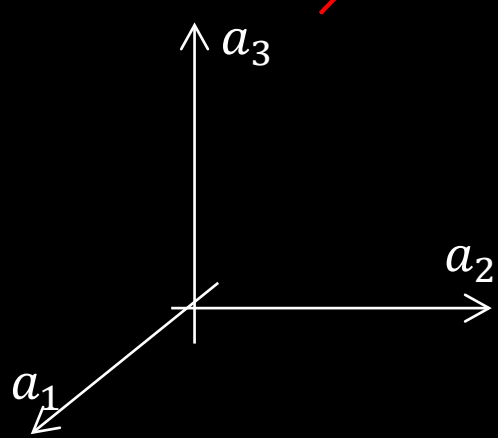
Rotation Matrix using Roll-Pitch-Yaw (X-Y-Z) – *How to Back out angles?*

$$R_{XYZ} = R_{x,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\phi s_\psi + s_\phi s_\theta c_\psi & c_\phi c_\psi - s_\phi s_\theta s_\psi & -s_\phi c_\theta \\ s_\phi s_\psi - c_\phi s_\theta c_\psi & s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\phi c_\theta \end{bmatrix}$$



$$R_D^A = R_B^A \times R_C^B \times R_D^C$$



Rotation Matrix using Roll-Pitch-Yaw (X-Y-Z) – *How to Back out angles?*

$$R_{XYZ} = R_{x,\phi} R_{y,\theta} R_{z,\psi}$$

$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\phi s_\psi + s_\phi s_\theta c_\psi & c_\phi c_\psi - s_\phi s_\theta s_\psi & -s_\phi c_\theta \\ s_\phi s_\psi - c_\phi s_\theta c_\psi & s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\phi c_\theta \end{bmatrix}$$

- But the solution to acos is not unique
- atan(x) returns $[-\pi/2, \pi/2]$
- Instead use atan2(adj,opp)* which returns $[-\pi, \pi]$
 - $\theta = \text{asin}(r_{13})$
 - $\phi = \text{atan2}(-r_{23}, r_{33})$
 - $\psi = \text{atan2}(-r_{12}, r_{11})$
- Special case if $r_{13}=1$ (the z' axis is parallel to the x-axis)
 - $\theta = r_{13} 90^\circ, \psi = \text{atan2}(r_{21}, r_{22}), \phi = 0^\circ$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

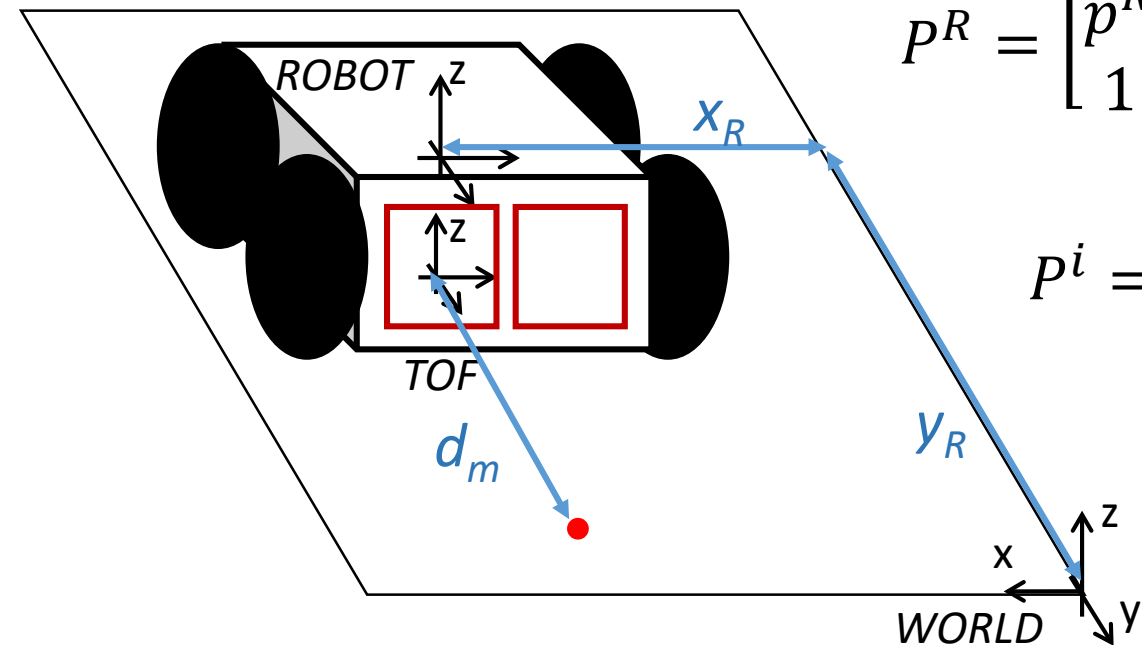
```
float atan2(float x, float y) {
    if (x > 0.0)
        return atan(y/x);
    if (x < 0.0) {
        if (y >= 0.0)
            return (PI + atan(y/x));
        else
            return (-PI + atan(y/x));
    }
    if (y > 0.0) // x == 0
        return PI_ON_TWO;
    if (y < 0.0)
        return -PI_ON_TWO;
    return 0.0; // Should be undefined
}
```

**These are not consistent across platforms!*

Homogeneous Transformation Matrix

$$T = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta & d_x \\ c_\phi s_\psi + s_\phi s_\theta c_\psi & c_\phi c_\psi - s_\phi s_\theta s_\psi & -s_\phi c_\theta & d_y \\ s_\phi s_\psi - c_\phi s_\theta c_\psi & s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\phi c_\theta & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotation
translation



$$p^R = \begin{bmatrix} p^R \\ 1 \end{bmatrix}$$

$$p^i = T_R^i p^R$$

$$p^i = T_R^i T_{TOF}^R p^{TOF}$$

$$p^i =$$

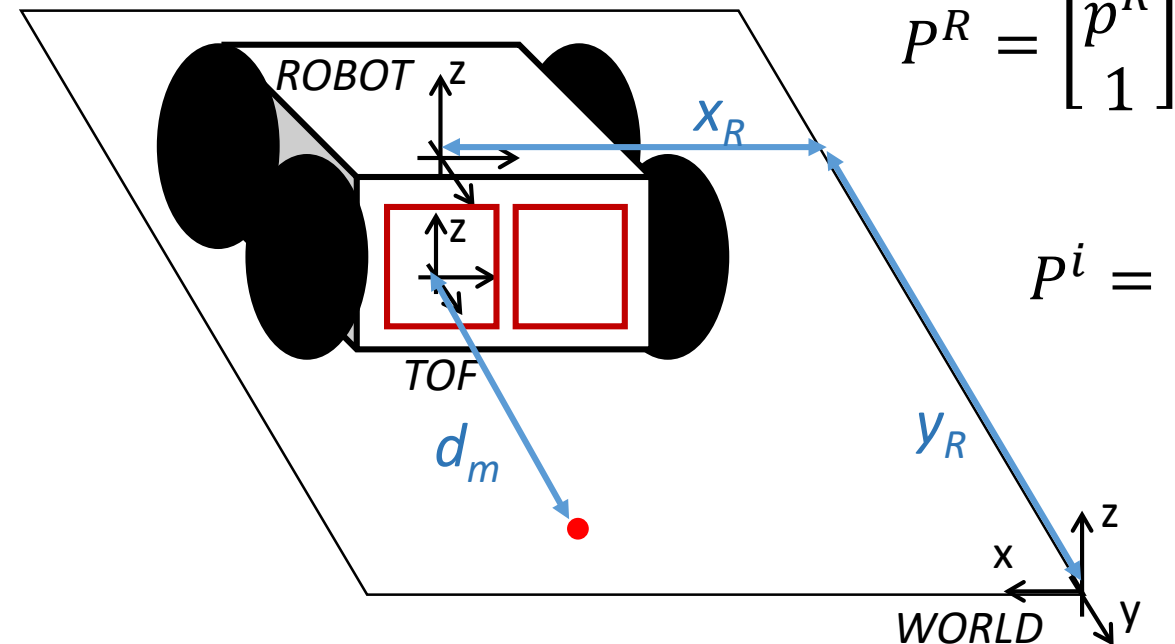
$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 0 & 1 & \end{bmatrix} \begin{bmatrix} d_m \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Homogeneous Transformation Matrix

$$T = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta & d_x \\ c_\phi s_\psi + s_\phi s_\theta c_\psi & c_\phi c_\psi - s_\phi s_\theta s_\psi & -s_\phi c_\theta & d_y \\ s_\phi s_\psi - c_\phi s_\theta c_\psi & s_\phi c_\psi + c_\phi s_\theta s_\psi & c_\phi c_\theta & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotation
translation



$$p^R = \begin{bmatrix} p^R \\ 1 \end{bmatrix}$$

$$p^i = T_R^i p^R$$

$$p^i = T_R^i T_{TOF}^R p^{TOF}$$

$$\begin{bmatrix} 0 & -1 & 0 & x_R \\ 1 & 0 & 0 & -y_R \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0.07 \\ 0 & 1 & 0 & -0.015 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_m \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

if $x_R = 1, y_R = 1, d_m = 1$:
 $= [1.015 \quad 0.07 \quad 0 \quad 1]^T$

Moving Frame versus Fixed Frame Rotation

- Equivalent, BUT the order matters
- Mobile frame
 - Post-multiply
- Fixed frame
 - Pre-multiply

<https://www.mecademic.com/resources/Euler-angles/Euler-angles>

Tutorial: How is orientation in space

mecademic.com/resources/Euler-angles/Euler-angles

Apps MPI Cornell Passkey ECE3400 CEI-Lab Slack Other bookmarks

MECADEMIC

x y z 32°

x y z 33°

x y z 27°

α : < 32 >

β : < 33 >

γ : < 27 >

[0.747 -0.128 0.652]

Sources and References

- Northwestern University, course on Modern Robotics
- Upenn Coursera course on Aerial Robotics
- MilfordRobotics youtube stream
- Mecademic

Logistics

- Send me your Github page link!
- Thursday is a “setup help” lecture
- PH427 is ready!
 - One person at a time
 - Book via Google calendar
 - Inform Campuswire
 - iSanitize & wear your mask!

