ECE 4960

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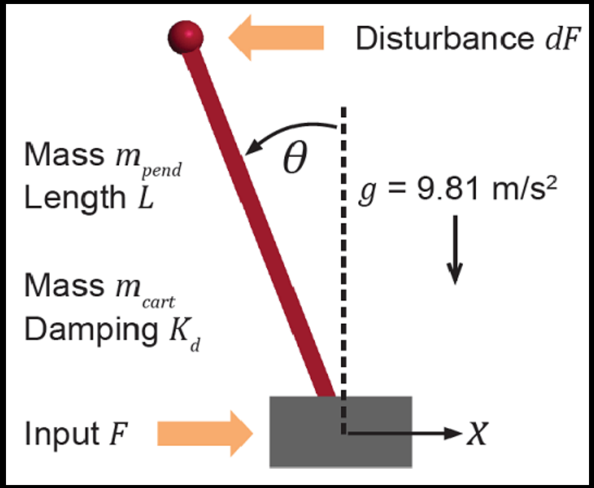
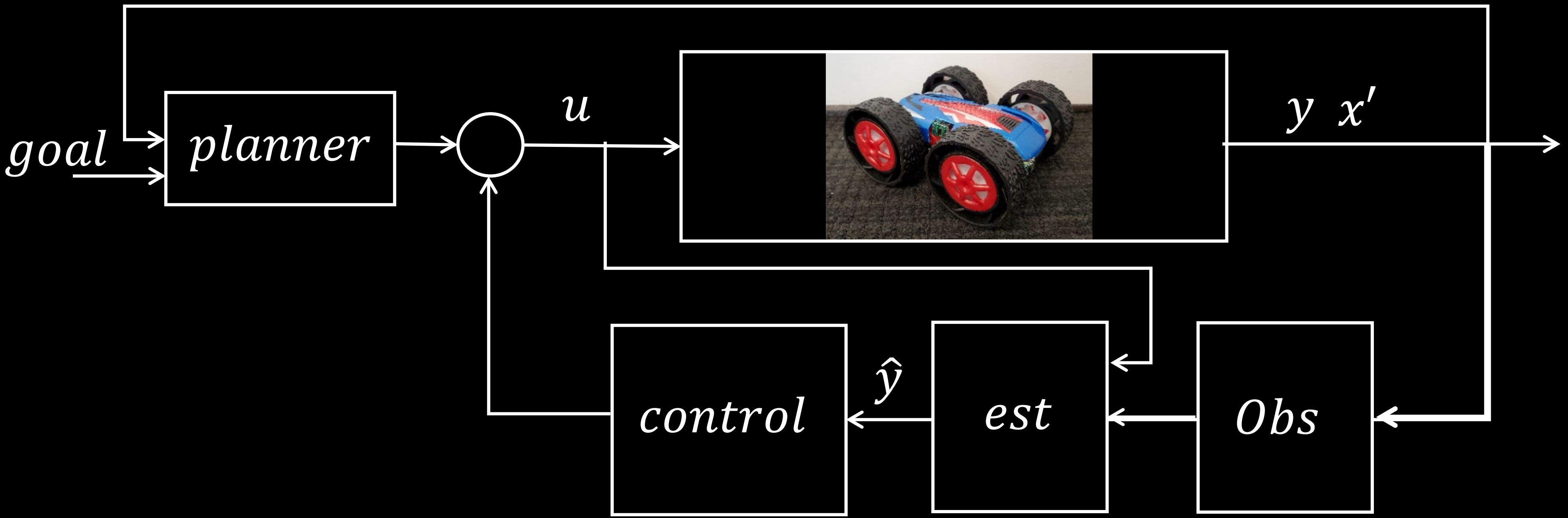
Fast Robots

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- Observability



$$\dot{x} = Ax + Bu$$



Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

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$$\dot{x} = Ax + Bu$$

This should look familiar from..
- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

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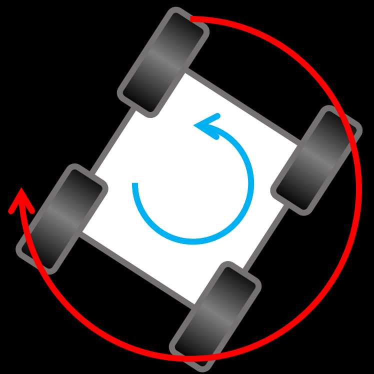
$$\dot{x} = Ax + Bu$$

Lecture 7-8 - PID



1st order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & \frac{-c}{I} \end{bmatrix}} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix}} u$

2nd order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ cst & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$

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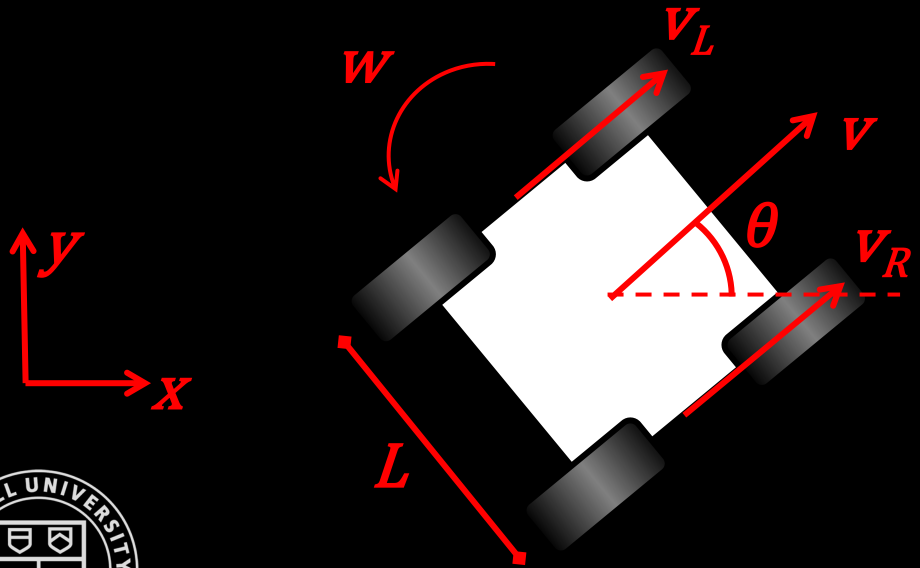
$$\dot{x} = Ax + Bu$$



$$\dot{x} = \cos(\theta)v$$
$$\dot{y} = \sin(\theta)v$$
$$\dot{\theta} = w$$

$$\Rightarrow \boxed{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos(\theta) & 0\\ \sin(\theta) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}}$$

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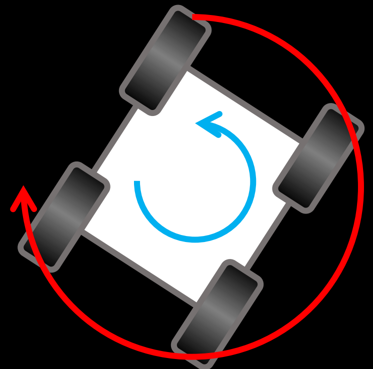

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2nd order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ cst & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$

Linear Systems

vector

matrix

Linear system $\dot{x} = Ax$ $\quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$

Basic solution $x(t) = e^{At} x(0)$

$$\left({}^{dx}/_{dt} = kx \leftrightarrow {}^{dx}/_{x} = kdt \leftrightarrow \ln(|x|) = kt + c \atop |x| = e^{kt} + e^c = \pm c e^{kt} \right)$$

Taylor series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

Intuition for eigenvectors (and stability)…

Linear system $$\underbrace{\dot{x}}_{\text{vector}} = \underbrace{A}_{\text{matrix}}x$$

Basic solution $$x(t) = e^{At}x(0)$$

Map the system to eigenvector coordinates to make computation easier

- Apply a linear transform: $z = Tx \quad \Leftrightarrow x = T^{-1}z$
- Substitute into the original equation: $T^{-1}\dot{z} = AT^{-1}z \quad \Leftrightarrow \dot{z} = TAT^{-1}z$
- Pick the matrix, T, such that TAT^{-1} becomes simpler than A

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Eigenvectors

Eigenvectors and Eigenvalues

matrix

vector

$$A\xi = \lambda\xi$$

scalar number (eigenvalue)

- Eigenvectors, ξ, of A
- Matrix of eigenvectors, T

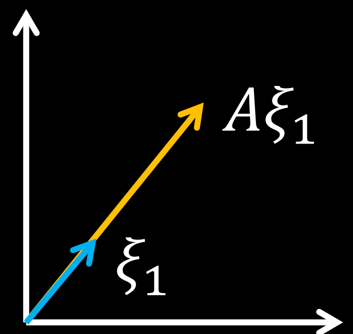
$$T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]^{*}$$

- Diagonal matrix of eigenvalues, D

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\boxed{AT = TD}$$

- $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$
- $\xi_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\lambda_1 = 4$



z-coordinates versus x-coordinates

$$\dot{x} = Ax \qquad x = \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$AT = TD \qquad \Leftrightarrow T^{-1}AT = D$$

Matlab $\gg [T, D] = eig(A);$

in the eigenvector directions

$$x = Tz$$

$$\dot{x} = T\dot{z} \quad = Ax$$

$$T\dot{z} = ATz$$

$$\dot{z} = T^{-1}ATz$$

$$\boxed{\dot{z} = Dz}$$

$\rightarrow$ By mapping our system to eigenvector coordinates, the dynamics become diagonal (very simple!)

z-coordinates versus x-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$T^{-1}AT = D$$

$$\dot{z} = Dz$$

Choose T appropriately, and you can usually get A into the diagonal form

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$z_1(t) = e^{\lambda_1 t} z_1(0) \quad \dots \quad z_n(t) = e^{\lambda_n t} z_n(0)$$

$$z(t) = e^{Dt} z(0) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} z(0)$$

…it is much simpler to think about your system in eigenvector coordinates!

z-coordinates versus x-coordinates

$$\boxed{\begin{aligned} \dot{x} &= Ax = T\dot{z} \\ T^{-1}AT &= D \\ \dot{z} &= Dz \end{aligned}}$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = TT^{-1} + TDT^{-1}t + \underbrace{(TD\underbrace{T^{-1}T}_{I}DT^{-1})}_{TD^2T^{-1}}\frac{t^2}{2!} + \ldots$$

z-coordinates versus x-coordinates

$$\boxed{\begin{aligned}\dot{x} &= Ax = T\dot{z}\\ T^{-1}AT &= D\\ \dot{z} &= Dz\end{aligned}}$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = TT^{-1} + TDT^{-1}t + (TDT^{-1}TDT^{-1})\frac{t^2}{2!} + \ldots$$

$$e^{At} = T\left[I + Dt + \frac{D^2t^2}{2!} + \cdots + \frac{D^nt^n}{n!}\right]T^{-1} = Te^{Dt}T^{-1}$$

easy to compute!

z-coordinates versus x-coordinates

$$\dot{x} = Ax$$

$$x(t) = e^{At}x(0)$$

$$AD = TD$$

$$x = Tz$$

system solution in physical coordinates

$$x(t) = \underbrace{T\underbrace{e^{Dt}\underbrace{T^{-1}x(0)}_{z(0)}}_{z(t)}}_{x(t)}$$

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Eigenvalues and Stability

(very compressed version of full classes on ODEs)

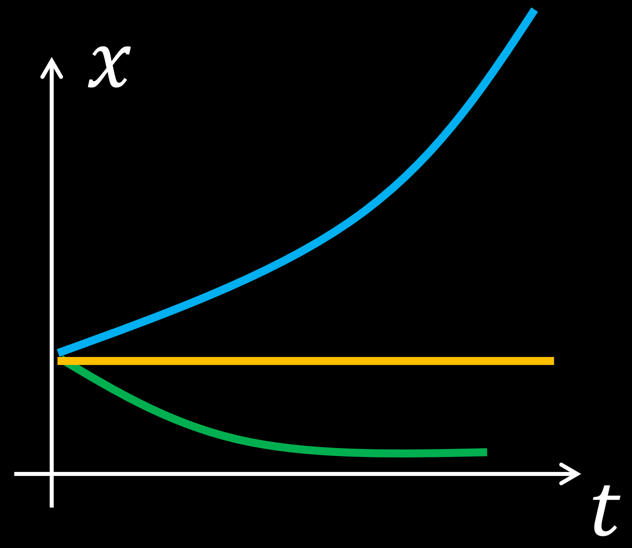
Stability (Continuous Time)

$$\dot{x} = Ax$$

$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$\gg [T, D] = la.eig(A);$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

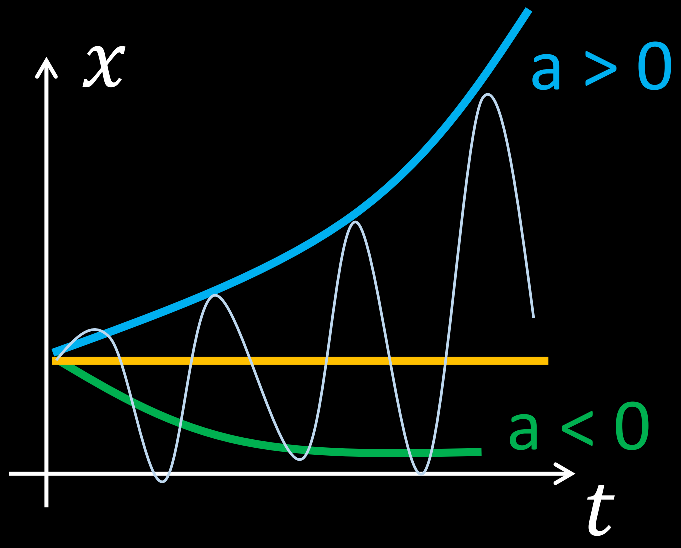


- If even one of the $e^{\lambda_n t}$ goes to ∞ all go to ∞
- Complex eigenvalues
  - $\lambda = a + ib$
- Euler's formula
  - $e^{\lambda t} = e^{at}[\cos(bt) + i\sin(bt)]$

Stability (Continuous Time)

$$\dot{x} = Ax$$
$$x(t) = Te^{Dt}T^{-1}x(0)$$
$$\gg [T, D] = la.eig(A);$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$



- If even one of the $e^{\lambda_n t}$ goes to ∞ all go to ∞
- Complex eigenvalues
  - $\lambda = a \pm ib$
- Euler's formula
  - $e^{\pm\lambda t} = e^{at}[\underbrace{\cos(bt) \pm i\sin(bt)}_{=1}]$

When is this stable?

- The system is stable iff all the real parts of all the eigenvalues are negative!

Stability (Continuous Time)

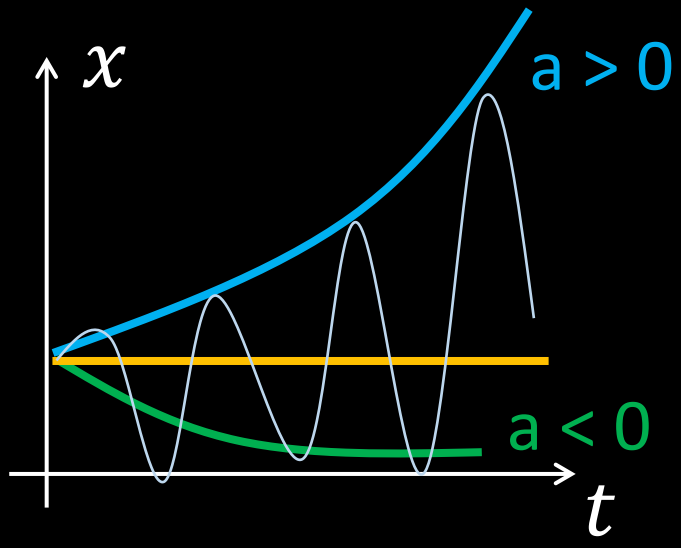
$$\dot{x} = Ax$$

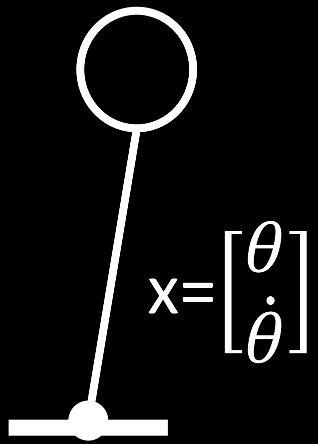
$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$\gg [T, D] = la.eig(A);$$

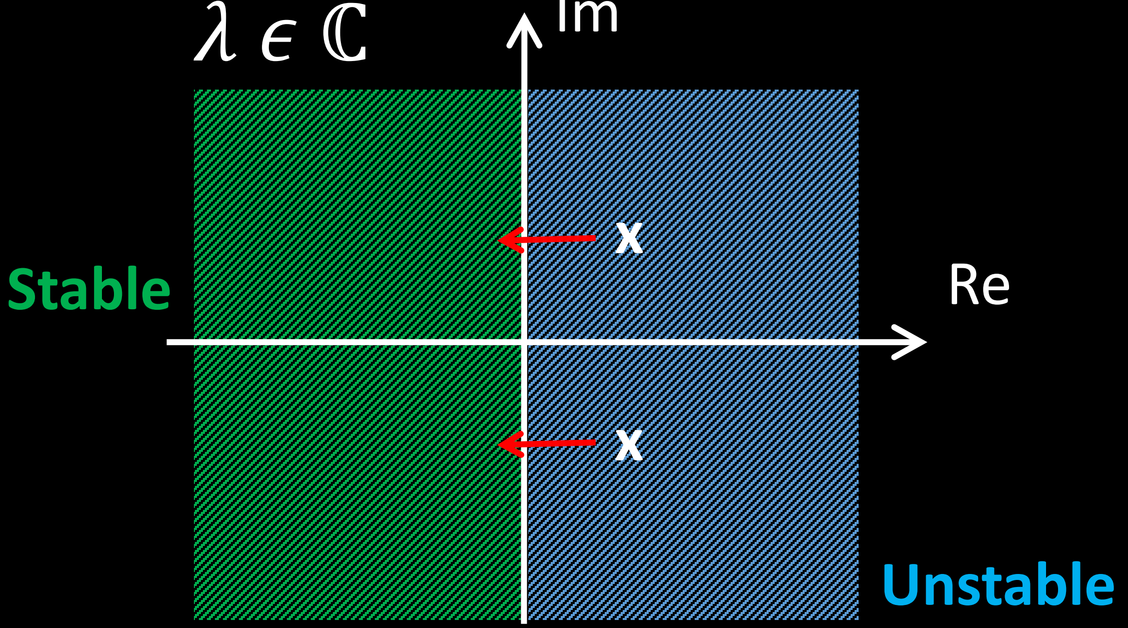
$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

$$\lambda = a + ib$$



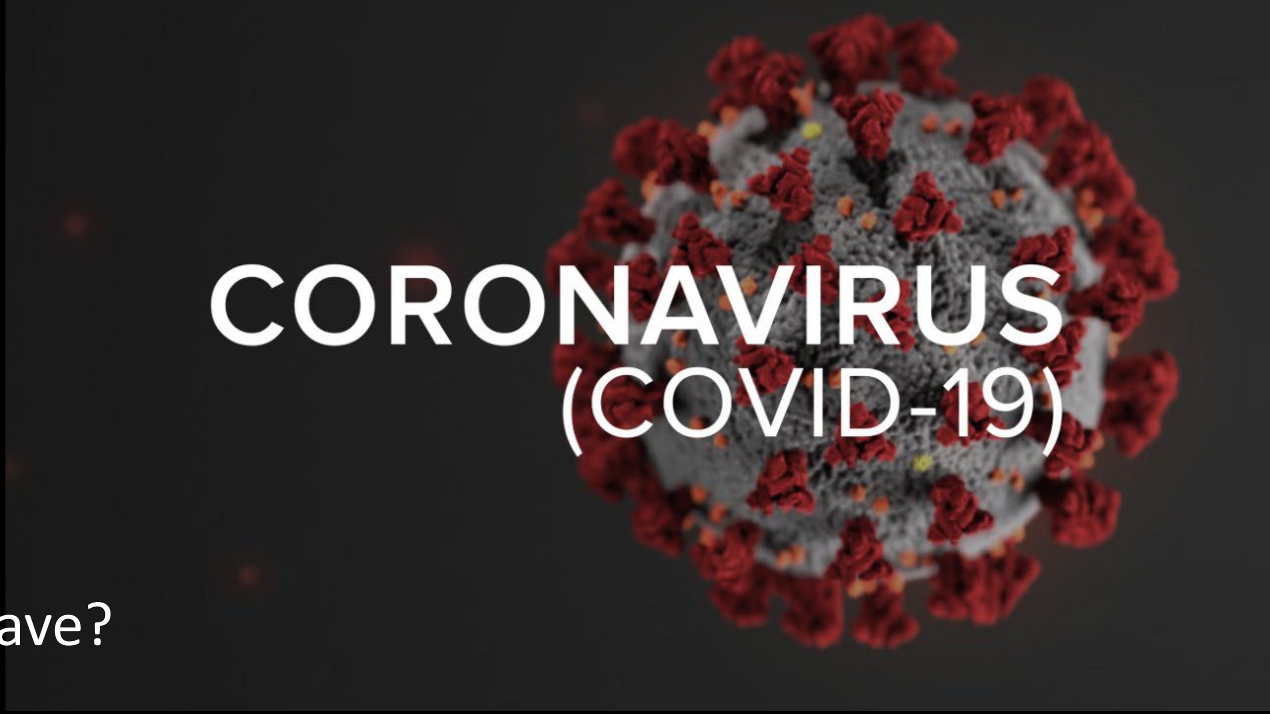


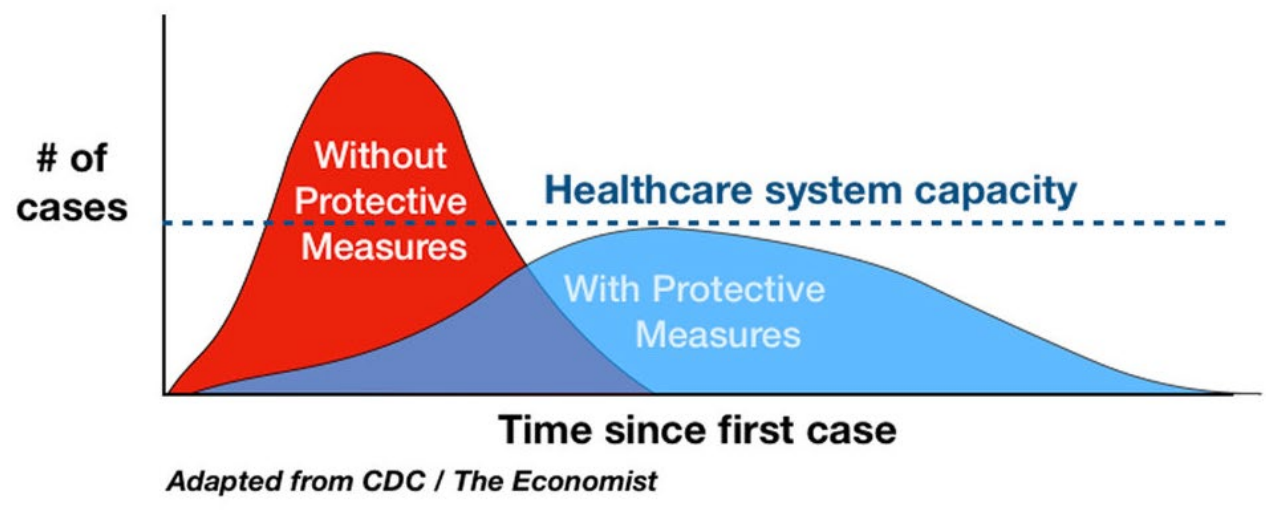
$\lambda \in \mathbb{C}$



Stability (Continuous Time)

- COVID-19, very simplified example
- What is our state vector, x?
  - $x = [\#infected\ people]$
- System
  - $\dot{x} = Ax$
- How many eigenvectors does the system have?
  - 1
- Is the system unstable?
  - The eigenvalue of A / A has a positive real part
- What are our control inputs?
  - Wearing masks
  - Social distancing
  - Vaccines
  - Heard immunity
  - Etc..





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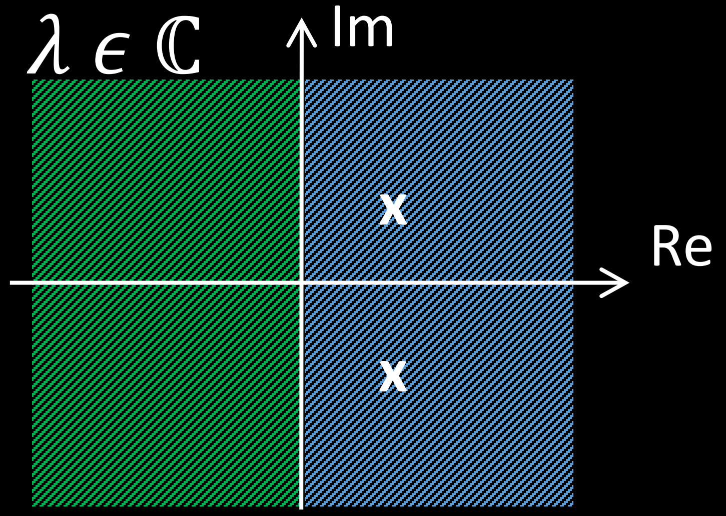
Discrete Time Systems

Stability (Discrete Time)

$\dot{x} = Ax$

$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$



Stable Unstable

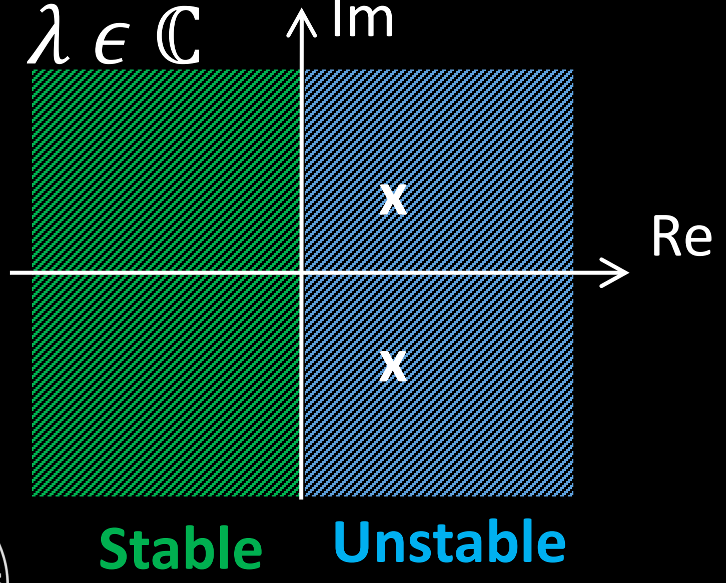
- $x(k+1) = \tilde{A}x(k) \quad , x(k) = x(k\Delta t)$
- How does $\tilde{A}$ relate to A? $\tilde{A} = e^{A\Delta t}$
- $x_1 = \tilde{A}x_0$ $\qquad \tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$ $\qquad \tilde{\lambda}$
- $x_2 = \tilde{A}x_1 = \tilde{A}^2x_0$ $\qquad \tilde{A}^2 = \tilde{T}\tilde{D}^2\tilde{T}^{-1}$ $\qquad \tilde{\lambda}^2$
- $x_3 = \tilde{A}^3x_0$ $\qquad \tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1}$ $\qquad \tilde{\lambda}^3$
- … $\qquad$ … $\qquad$ …
- $x_n = \tilde{A}^nx_0$ $\qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1}$ $\qquad \tilde{\lambda}^n$

Stability (Discrete Time)

$$\dot{x} = Ax$$

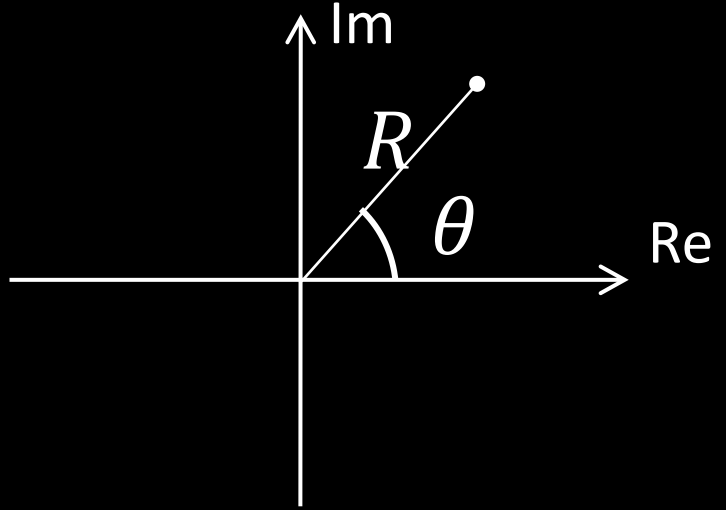
$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$



- $x(k+1) = \tilde{A}x(k) \quad , x(k) = x(k\Delta t)$
- $\tilde{A} = e^{A\Delta t}$
- $x_1 = \tilde{A}x_0 \qquad \tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1} \qquad \tilde{\lambda}$
- $x_2 = \tilde{A}x_1 = \tilde{A}^2 x_0 \qquad \tilde{A}^2 = \tilde{T}\tilde{D}^2\tilde{T}^{-1} \qquad \tilde{\lambda}^2$
- $x_3 = \tilde{A}^3 x_0 \qquad \tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1} \qquad \tilde{\lambda}^3$
- $\dots \qquad \dots \qquad \dots$
- $x_n = \tilde{A}^n x_0 \qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} \qquad \tilde{\lambda}^n$

- $\tilde{\lambda} = Re^{i\theta}$
- $\tilde{\lambda}^n = R^n e^{in\theta}$

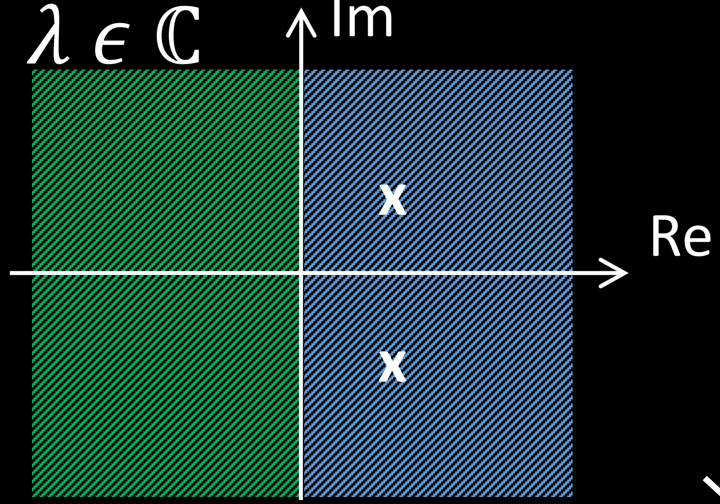


Stability (Discrete Time)

$$\dot{x} = Ax$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$

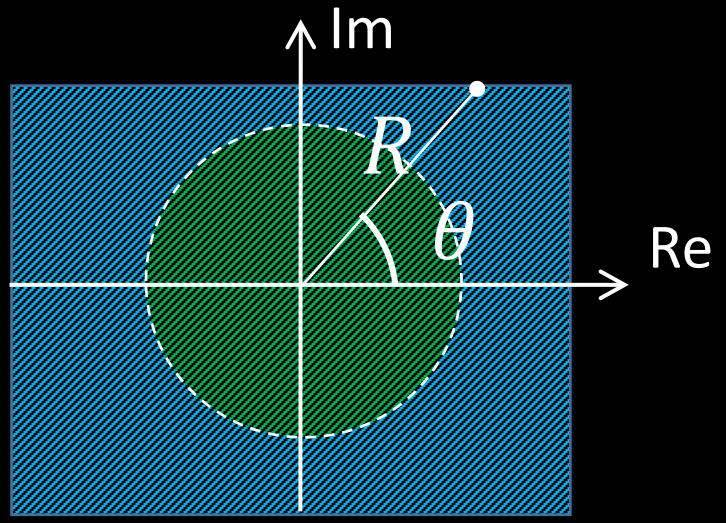


Stable **Unstable**

- $x(k+1) = \tilde{A}x(k) \quad , x(k) = x(k\Delta t)$
- $\tilde{A} = e^{A\Delta t}$
- $x_1 = \tilde{A}x_0 \qquad \tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1} \qquad \tilde{\lambda}$
- $x_2 = \tilde{A}x_1 = \tilde{A}^2x_0 \qquad \tilde{A}^2 = \tilde{T}\tilde{D}^2\tilde{T}^{-1} \qquad \tilde{\lambda}^2$
- $x_3 = \tilde{A}^3x_0 \qquad \tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1} \qquad \tilde{\lambda}^3$
- … … …
- $x_n = \tilde{A}^nx_0 \qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} \qquad \tilde{\lambda}^n$

- $\tilde{\lambda} = Re^{i\theta}$
- $\tilde{\lambda}^n = R^ne^{in\theta}$

$e^{A\Delta t}$



Stability (Discrete Time)

$$\dot{x} = Ax$$

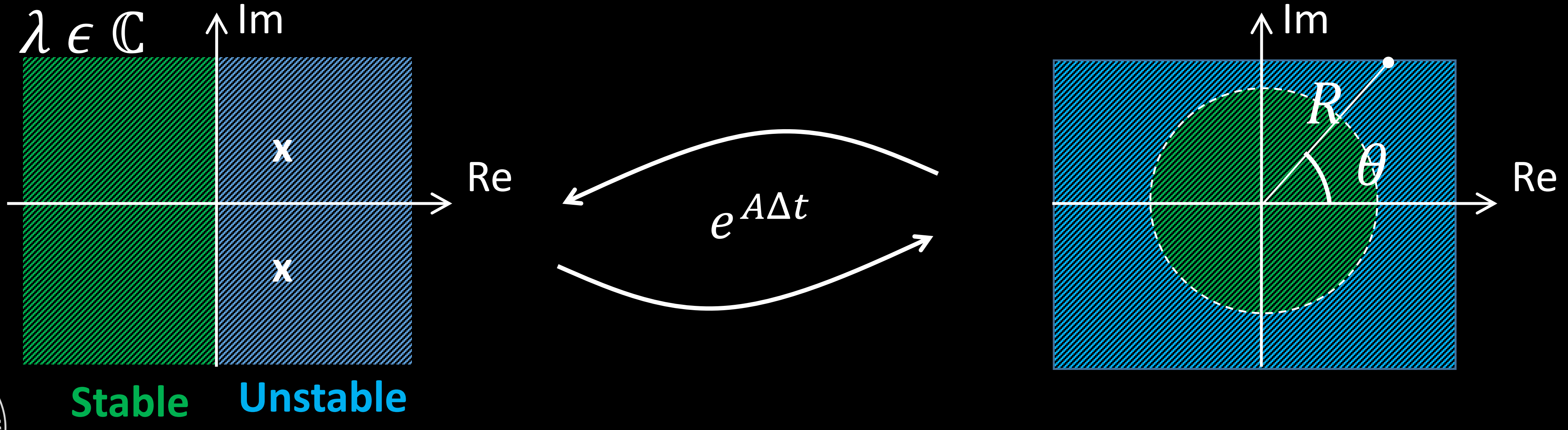
$$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$x(k+1) = \tilde{A}x(k)$$

$$\tilde{A} = e^{A\Delta t}$$

$$\tilde{\lambda}^n = R^n e^{in\theta}$$

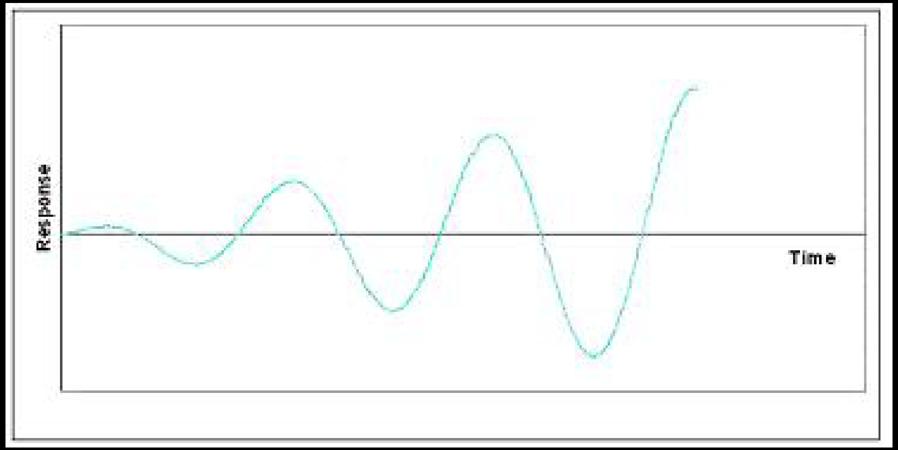
- We often work in discrete time
- Stability and quality of controllers depends on sampling rate

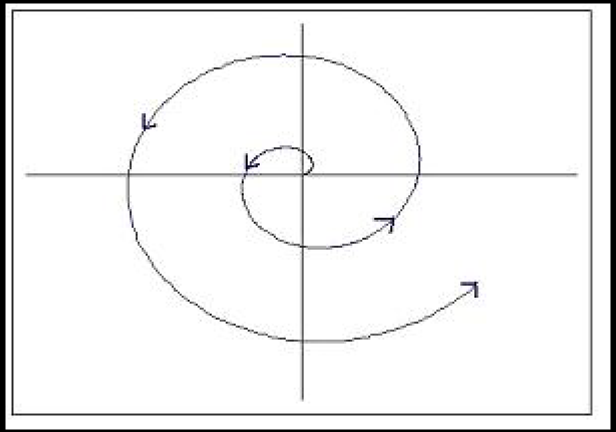


Stability (Discrete Time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k)$$



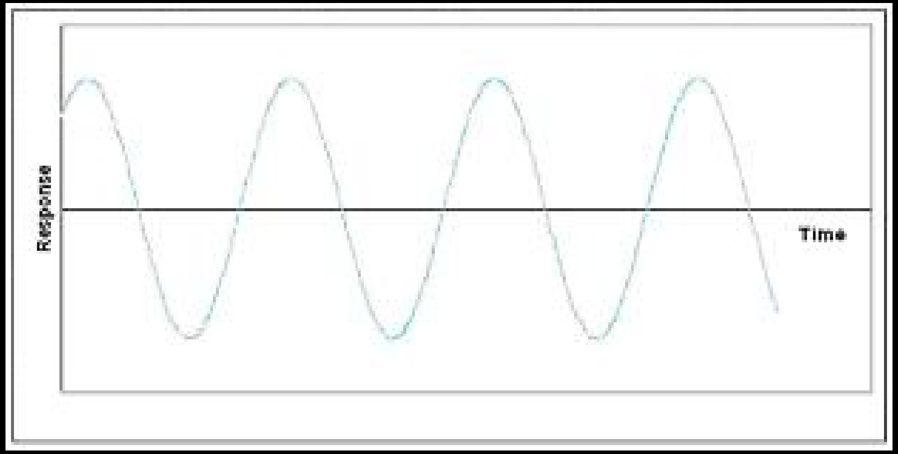


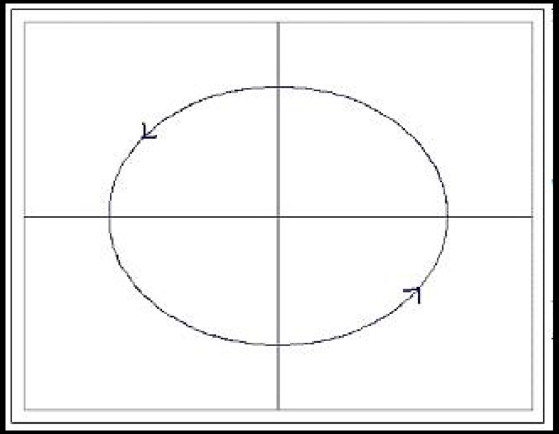
Unstable
(Positive real part)

Stability (Discrete Time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k)$$



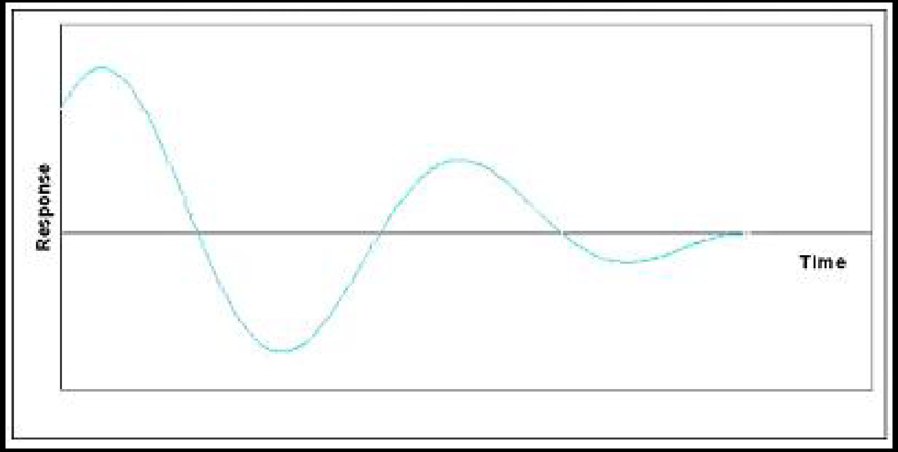


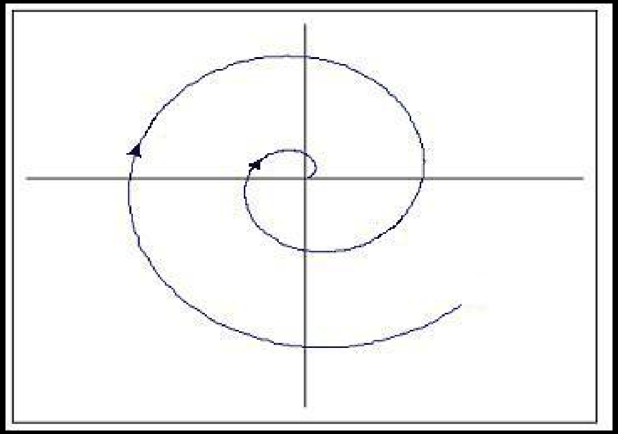
Critically stable
(Zero real part)

Stability (Discrete Time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k)$$





Stable
(Negative real part)

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Linearizing Nonlinear Systems

Linearizing Non-Linear Systems

Basic Steps to linearize a nonlinear system

1. Find some fixed points
   - $\bar{x}\ \ s.t.\ f(\bar{x}) = 0$
   - (basically points where the system doesn't move)
2. Linearize about $\bar{x}$
   - $\frac{Df}{Dx}\Big|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$ ← *"Jacobian"*

$$\dot{x} = f(x) \quad \Rightarrow \quad \dot{x} = Ax$$

Example

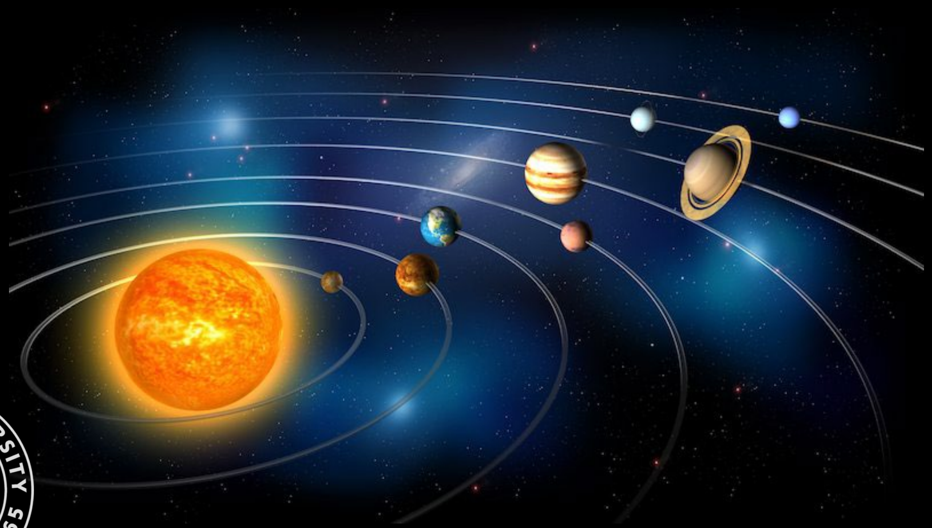
$$\dot{x_1} = f_1(x_1, x_2) = x_1 x_2$$

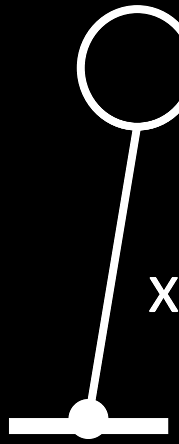
$$\dot{x_2} = f_2(x_1, x_2) = {x_1}^2 + {x_2}^2$$

$$\frac{Df}{Dx} = \begin{bmatrix} {\partial f_1}/{\partial x_1} & {\partial f_1}/{\partial x_2} \\ {\partial f_2}/{\partial x_1} & {\partial f_2}/{\partial x_2} \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$

Evaluate at $\bar{x}$





$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

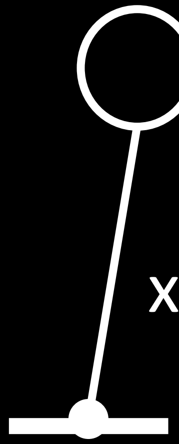
Linearizing Non-Linear Systems

$$\dot{x} = f(x) \quad \Rightarrow \quad \dot{x} = Ax$$

Basic Steps to linearize a nonlinear system

1. Find some fixed points
   - $\bar{x} \;\; s.t. \;\; f(\bar{x}) = 0$
   - (basically points where the system doesn't move)
2. Linearize about $\bar{x}$
   - $\frac{Df}{Dx}|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$ $\leftarrow$ *"Jacobian"*

- If you zoom in on $\bar{x}$, your system will look linear!



$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

Example

$$\dot{x_1} = f_1(x_1, x_2) = x_1 x_2$$

$$\dot{x_2} = f_2(x_1, x_2) = {x_1}^2 + {x_2}^2$$

$$\frac{Df}{Dx} = \begin{bmatrix} {\partial f_1}/{\partial x_1} & {\partial f_1}/{\partial x_2} \\ {\partial f_2}/{\partial x_1} & {\partial f_2}/{\partial x_2} \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$

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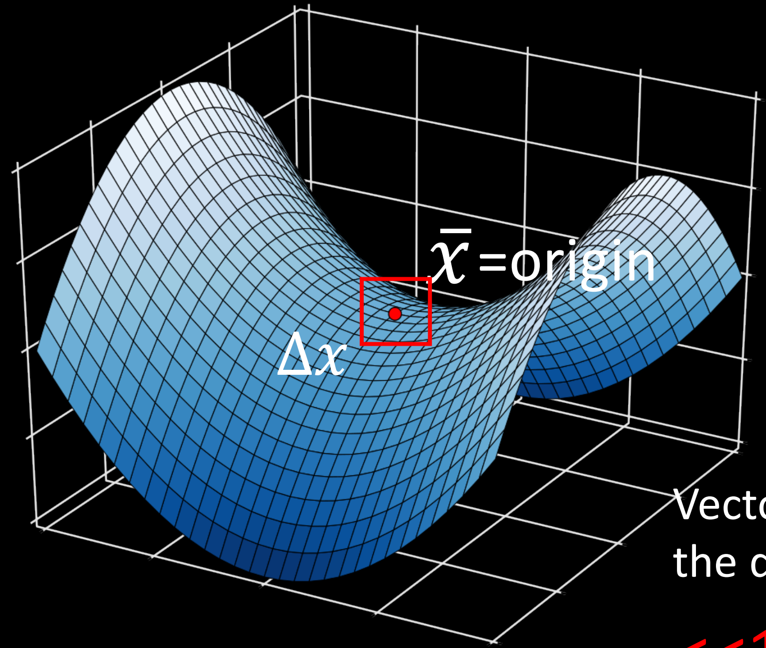

Vector field of the dynamics

$$\dot{x} = f(x)$$

$$\dot{x} = \overbrace{f(\bar{x})}^{0} + \frac{Df}{Dx}|_{\bar{x}}(x - \bar{x}) + \overbrace{\frac{D^2 f}{D^2 x}|_{\bar{x}}(x - \bar{x})^2 + \frac{D^3 f}{D^3 x}|_{\bar{x}}(x - \bar{x})^3 + \cdots}^{\ll 1}$$

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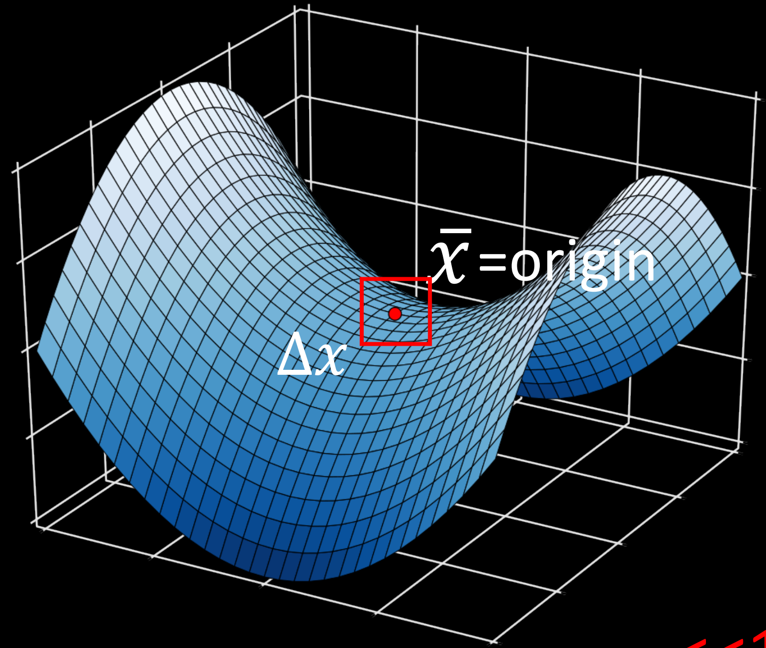

- Good control will keep you close to the fixed point, where your model is valid!

$$\dot{x} = f(x)$$

$$\dot{x} = \overbrace{f(\bar{x})}^{0} + \frac{Df}{Dx}|_{\bar{x}}(x-\bar{x}) + \overbrace{\frac{D^2 f}{D^2 x}|_{\bar{x}}(x-\bar{x})^2 + \frac{D^3 f}{D^3 x}|_{\bar{x}}(x-\bar{x})^3 + \cdots}^{<<1}$$

$$\Delta\dot{x} = \frac{Df}{Dx}|_{\bar{x}}\Delta x$$

$$\boxed{\Rightarrow \Delta\dot{x} = A\Delta x}$$

Linearizing Non-Linear Systems

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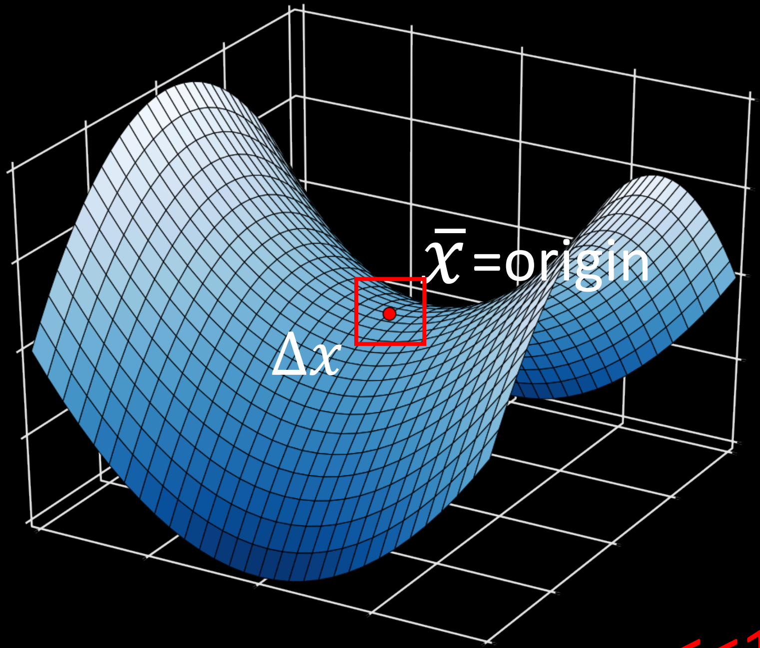

$$\dot{x} = f(x)$$

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Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$
- Linear transform: $AT = TD$
- Solution: $e^{At} = Te^{Dt}T^{-1}$
- Mapping from z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff a<0
  - Discrete time: $x(k+1) = \tilde{A}x(k)$, $\tilde{A} = e^{A\Delta t}$
    - Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff R<1

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>>[T,D] = eig(A)
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- Non-linear systems: $\dot{x} = f(x)$
- Linearization: $\frac{Df}{Dx}|_{\bar{x}}$