Prof. Kirstin Hagelskjær Petersen
kirstin@cornell.edu

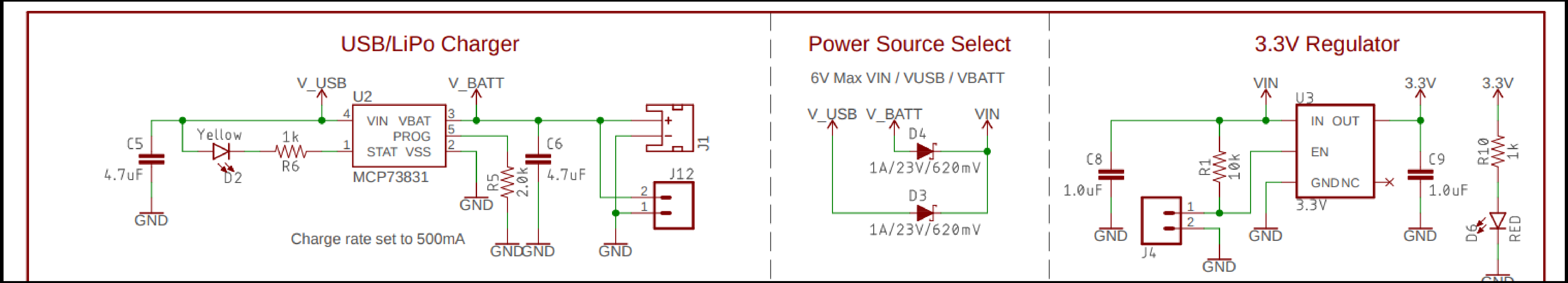
ECE 4160/5160
MAE 4910/5910

Fast Robots
Controllability

EdDiscussion posts…

Important posts in the recent days…

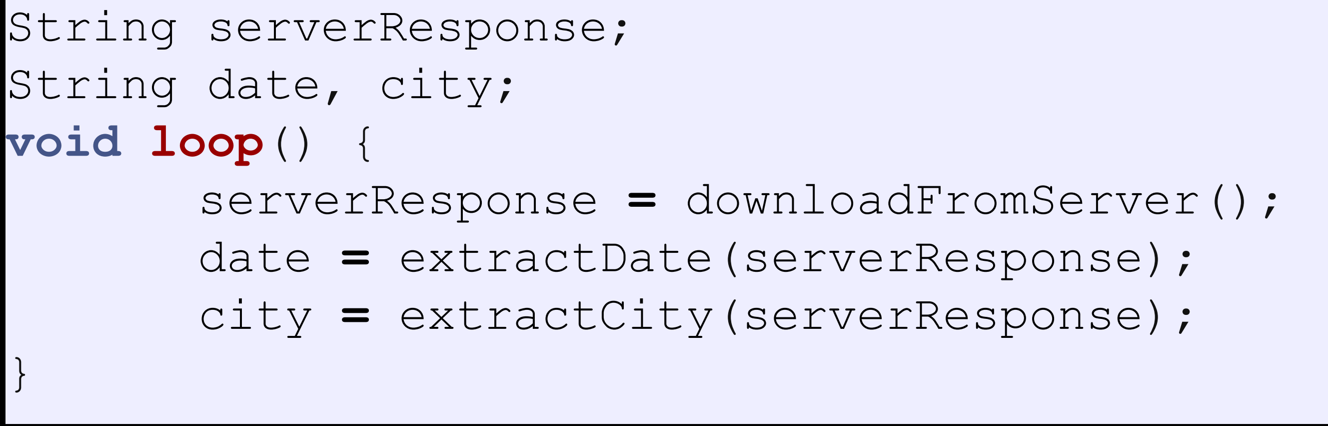
- Post #74 (Solution example for lab 5)
- Post #77 (Battery)



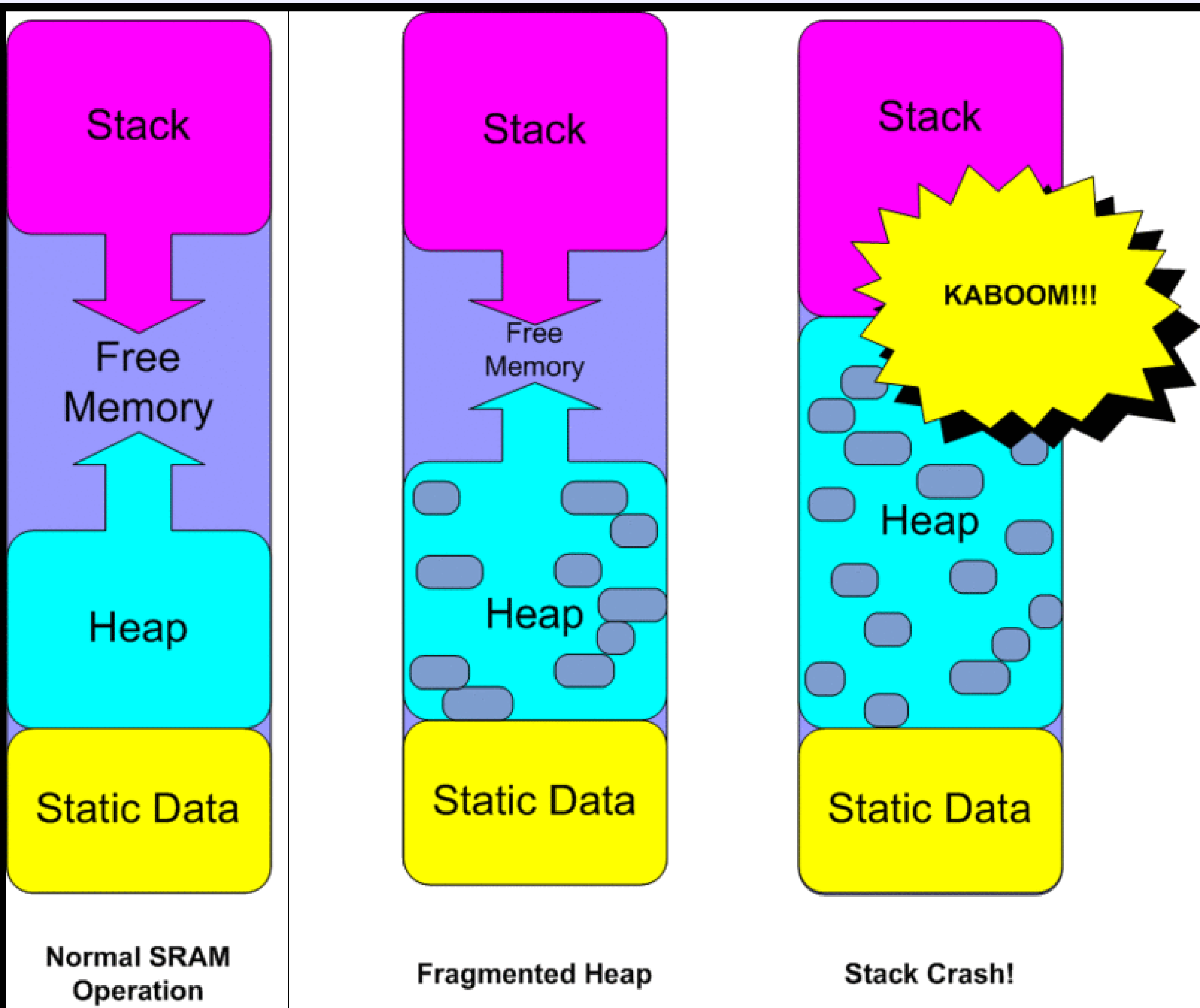
EdDiscussion posts…

Important posts in the recent days…

- Post #74 (Solution example for lab 5)
- Post #77 (Battery)
- Post #78 (memory)
  - FLASH
    - Arduino IDE reports this
  - SRAM
    - Static data/global variables
    - Heap (dynamically allocated data)
    - Stack (local variables, interrupt calls, function calls)
    - Allocate fixed memory
      - `myString.reserve(100);`
    - You can check: freeMemory()



```
String serverResponse;
String date, city;
void loop() {
        serverResponse = downloadFromServer();
        date = extractDate(serverResponse);
        city = extractCity(serverResponse);
}
```



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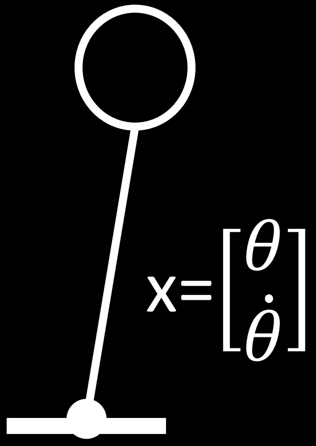
Fast Robots
Controllability

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- LQR
- Observability

$$\dot{x} = Ax + Bu$$



$$x=\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}$$

This should look familiar from..

- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

Linear Systems – “review of review”

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$

  `>>[T,D] = eig(A)`
- Linear transform: $AT = TD$
- Solution: $e^{At} = Te^{Dt}T^{-1}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff a<0
  - Discrete time: $x(k+1) = \tilde{A}x(k), \tilde{A} = e^{A\Delta t}$
    - Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff R<1
      - Method for linearizing non-linear systems

Linear Systems

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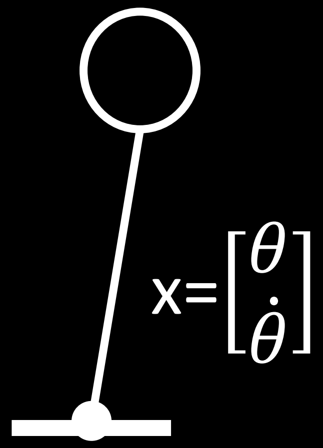

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Linearizing Nonlinear Systems

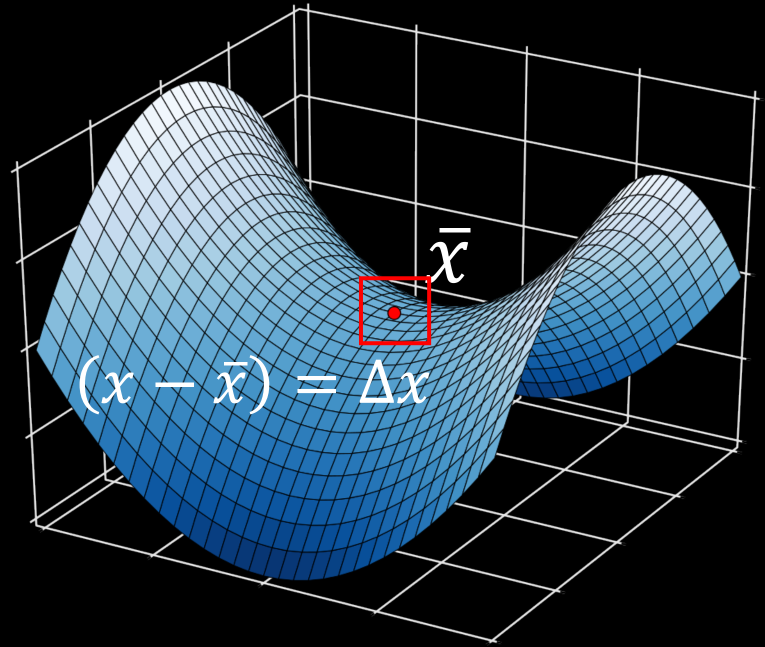
Linearizing Non-Linear Systems

$$\dot{x} = f(x) \qquad \Rightarrow \; \dot{x} = Ax$$

Basic Steps to linearize a nonlinear system

1. Find some fixed points
    - $\bar{x} \;\; s.t. \;\; f(\bar{x}) = 0$
    - (basically, points where the system doesn't move)
2. Linearize about $\bar{x}$
    - $\frac{Df}{Dx}|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$ ← *"Jacobian"*

- If you zoom in on $\bar{x}$, your system will look linear!

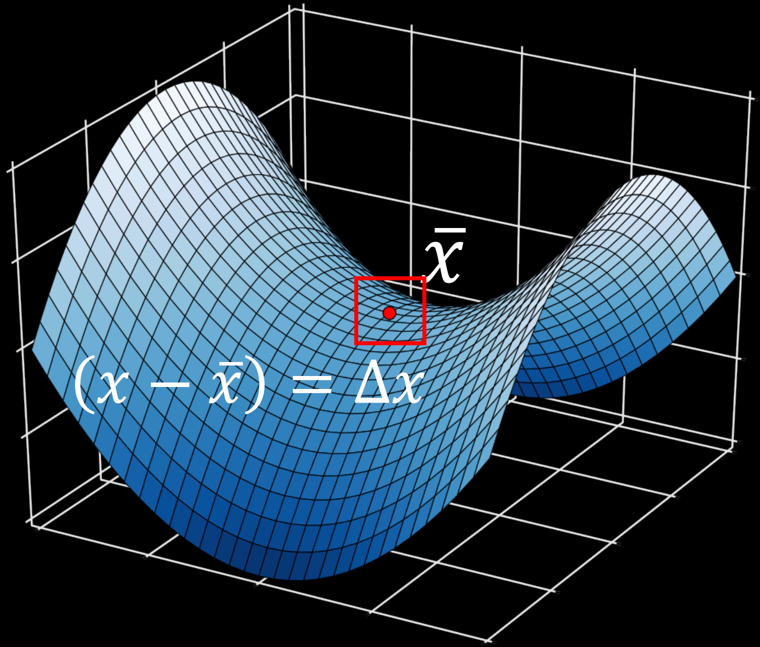


$$(x \,\dot{-}\, \bar{x}) = f(\bar{x}) + \frac{Df}{Dx}|_{\bar{x}}(x - \bar{x}) + \frac{D^2 f}{D^2 x}|_{\bar{x}}(x - \bar{x})^2 + \frac{D^3 f}{D^3 x}|_{\bar{x}}(x - \bar{x})^3 + \cdots$$

$$\boxed{\Delta\dot{x} = \frac{Df}{Dx}|_{\bar{x}}\Delta x \quad \Rightarrow \Delta\dot{x} = A\Delta x}$$

Linearizing Non-Linear Systems

$$\dot{x} = f(x) \quad \Rightarrow \quad \dot{x} = Ax$$



Basic Steps to linearize a nonlinear system

1. Find some fixed points
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2. Linearize about $\bar{x}$
   - $\frac{Df}{Dx}|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$ ← *"Jacobian"*

- If you zoom in on $\bar{x}$, your system will look linear!
- Good control will keep you close to the fixed point, where your model is valid!

$$\boxed{\Delta\dot{x} = \frac{Df}{Dx}|_{\bar{x}}\Delta x \quad \Rightarrow \Delta\dot{x} = A\Delta x}$$

Linearizing Non-Linear Systems

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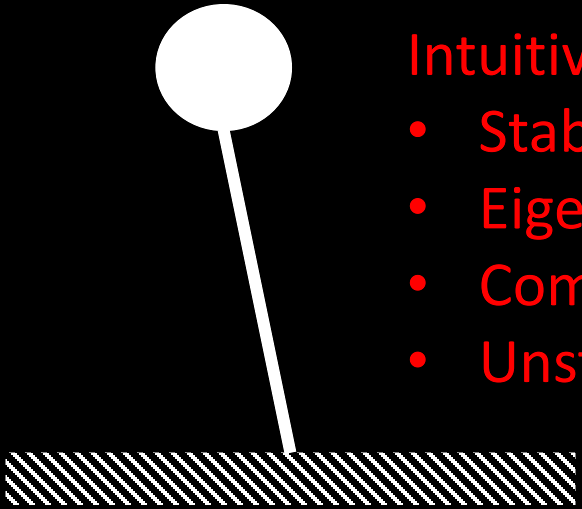
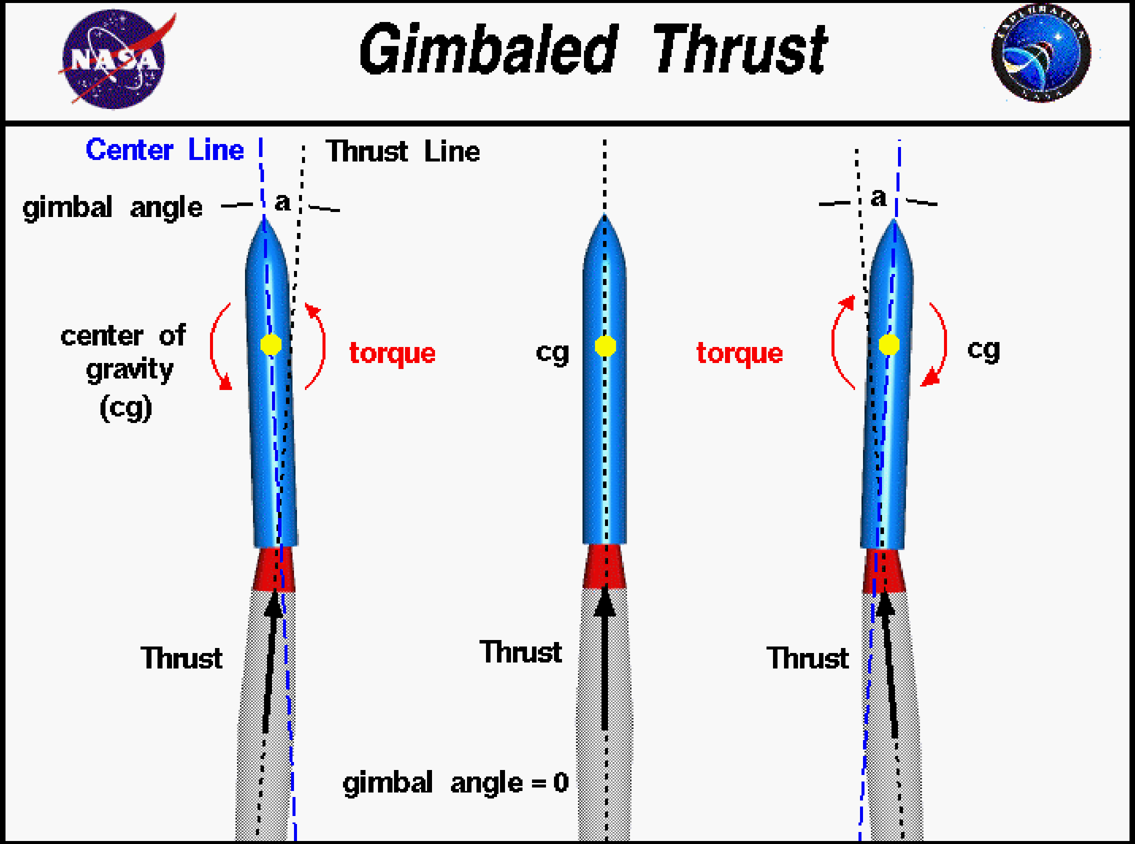


Intuitively, you know:

- Stable point
- Eigenvalues
- Complex poles
- Unstable point



Linearizing Non-Linear Systems

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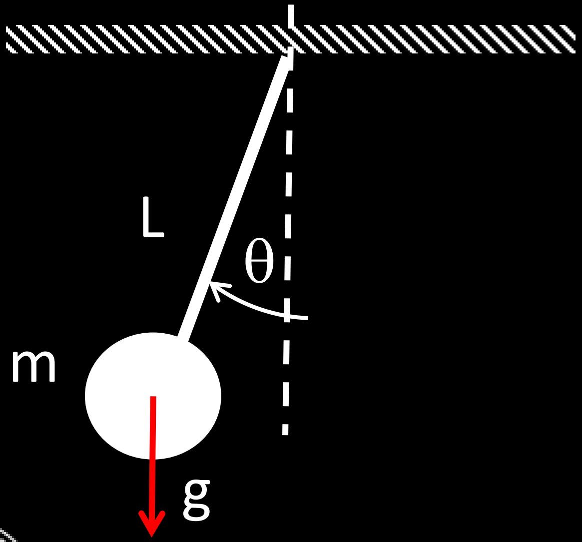

$$\dot{x} = f(x) \Rightarrow \qquad \dot{x} = Ax$$

Eq. of motion

- $\tau = -mgLsin(\theta)$
- $\tau = I\ddot{\theta}$
- $I\ddot{\theta} = -mgLsin(\theta)$
- Point mass inertia
  - $I = mL^2$
- $mL^2\ddot{\theta} = -mgLsin(\theta)$
- $\boxed{\ddot{\theta} = -\frac{g}{L}\sin(\theta)\ -\delta\dot{\theta}}$

friction

Linearizing Non-Linear Systems

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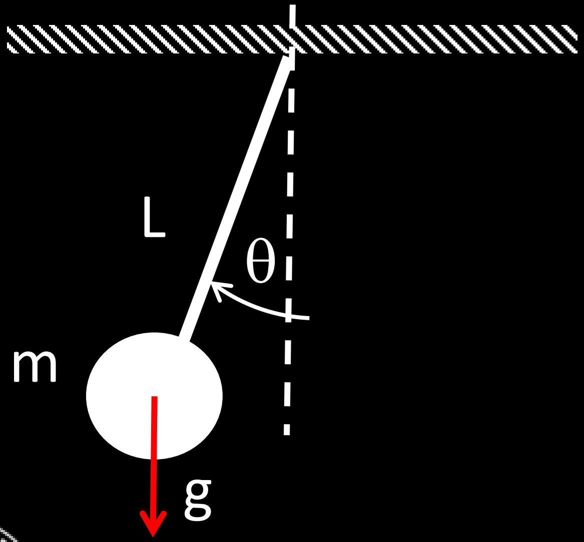
   - $\frac{Df}{Dx}|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$



$$\dot{x} = f(x) \Rightarrow \qquad \dot{x} = Ax$$

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \delta\dot{\theta}, \qquad \frac{g}{L} = 1$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

These points have physical meaning!

Linearizing Non-Linear Systems

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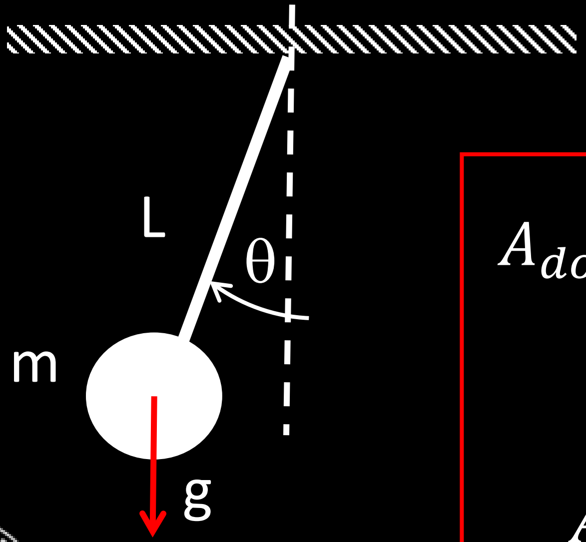
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0, \pi \\ 0 \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} {\partial f_1}/{\partial x_1} & {\partial f_1}/{\partial x_2} \\ {\partial f_2}/{\partial x_1} & {\partial f_2}/{\partial x_2} \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -\delta \end{bmatrix}$$



$$A_{down} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$$

$$A_{up} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}$$

Linearizing Non-Linear Systems

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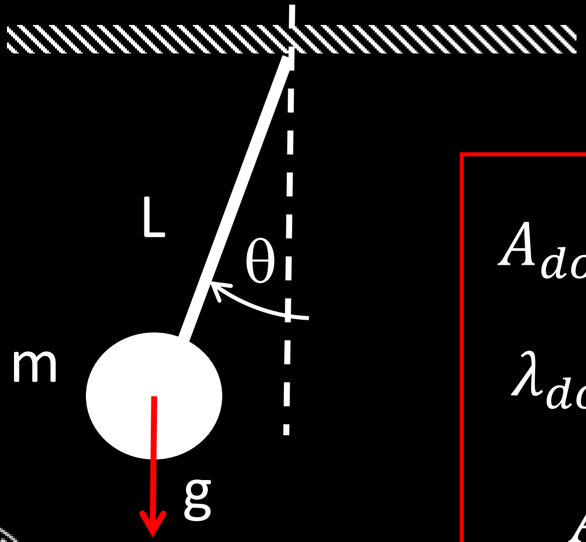

$$A_{down} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$$

$\lambda_{down} \approx -\delta' \pm i$ stable!

$$A_{up} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}$$

$\lambda_{up} \approx \pm 1$ unstable!

Linearizing Non-Linear Systems

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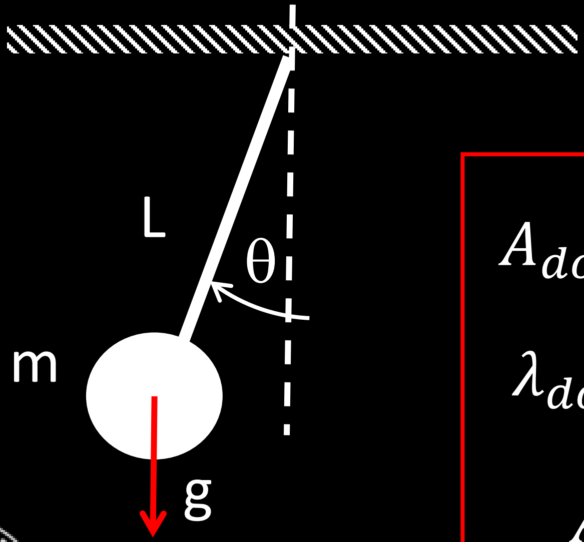

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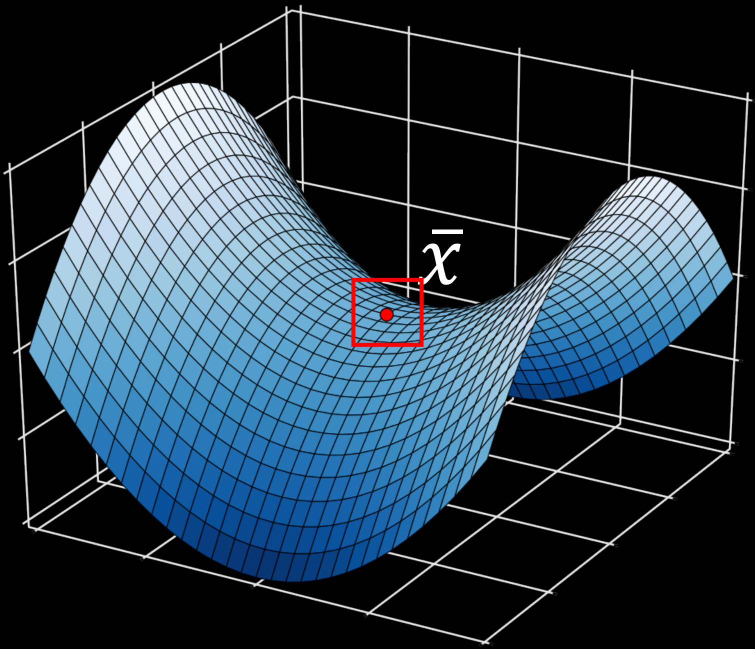

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Controllability

Controllability

- Is the system controllable?
- How do we design the control law, u?

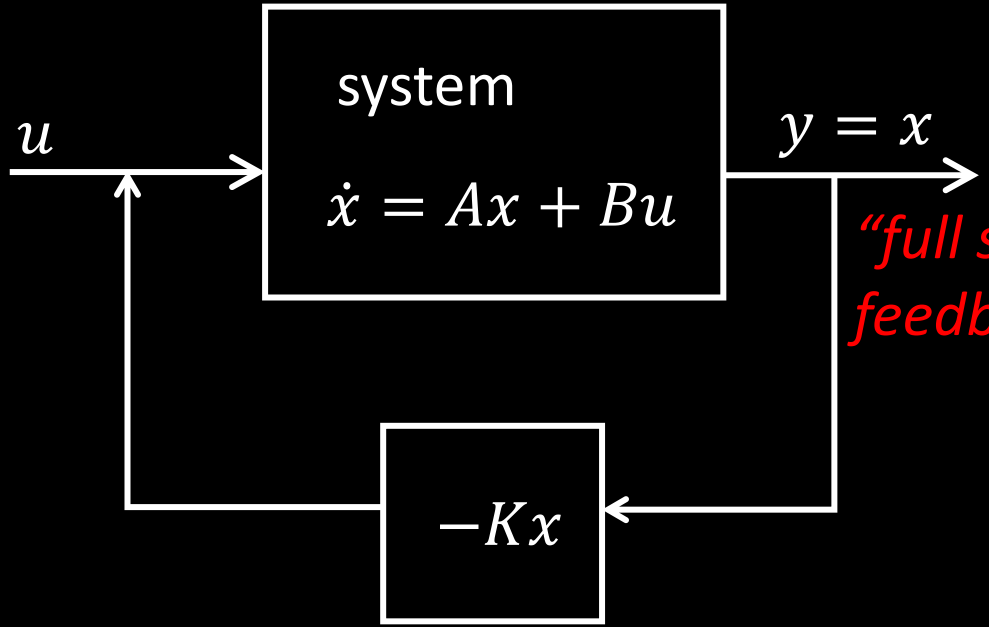
$$\dot{x} = Ax + Bu$$

$x \epsilon \mathbb{R}^n$

$\mathrm{A} \epsilon \mathbb{R}^{n \times m}$

$\mathrm{u} \epsilon \mathbb{R}^q$

$\mathrm{B} \epsilon \mathbb{R}^{n \times q}$



u

system

$\dot{x} = Ax + Bu$

$y = x$

$-Kx$

"full state feedback"

A linear controller (K matrix) can be optimal for linear systems!

Controllability

- Is the system controllable?
- How do we design the control law, u?

$$\dot{x} = Ax + Bu$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

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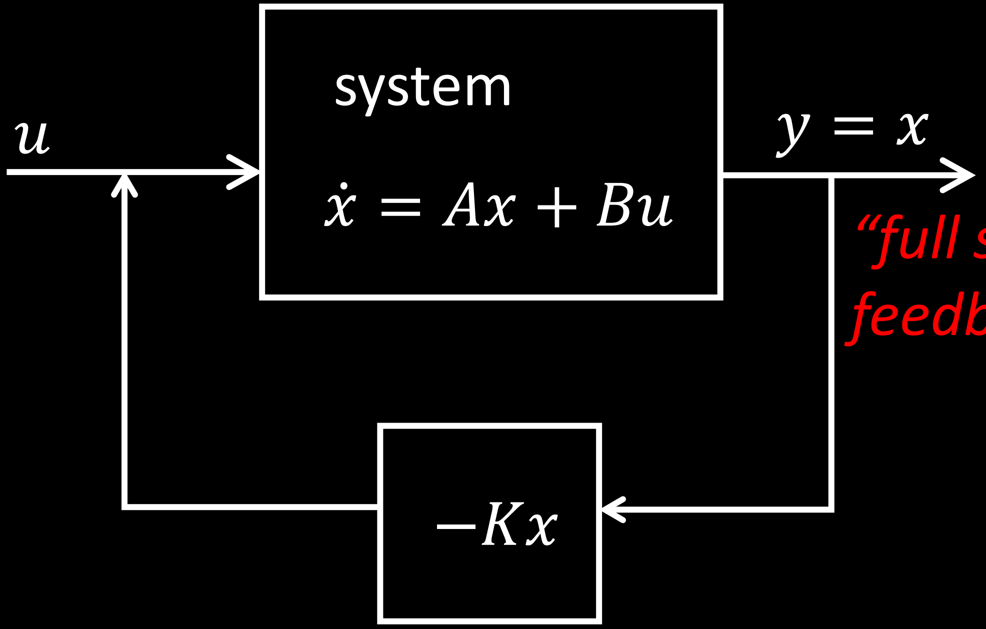

"full state feedback"

A linear controller (K matrix) can be optimal for linear systems!

Controllability

- What determines whether or not a system is controllable?
  - A system is controllable, if you can steer your state x anywhere you want in $\mathbb{R}^n$

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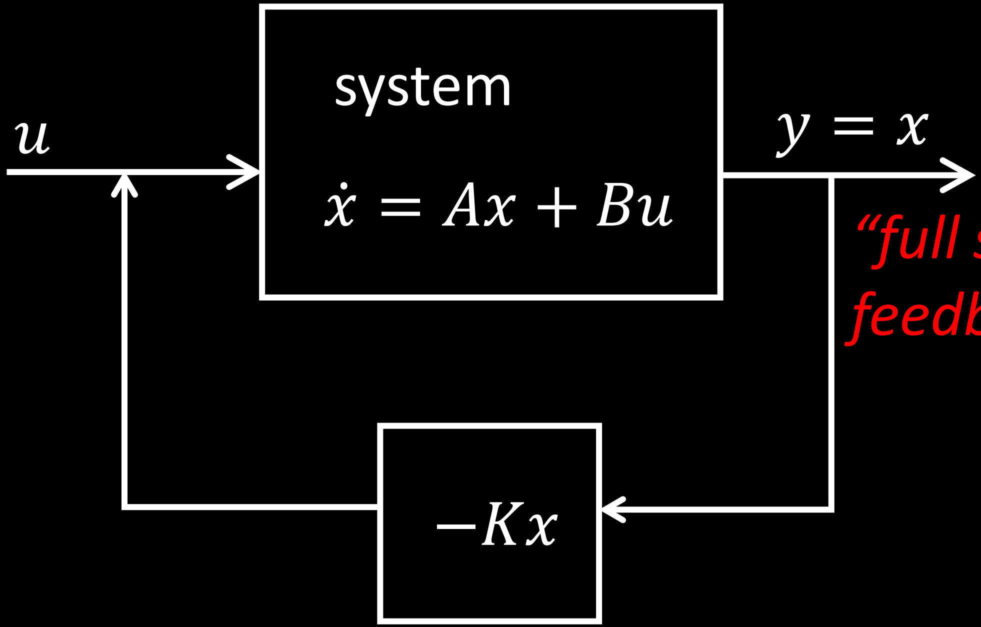

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Often, you don't get to choose A or B

Controllability

- What determines whether or not a system is controllable?
  - A system is controllable, if you can steer your state x anywhere you want in $\mathbb{R}^n$
  - Matlab >> rank(ctrb(A,B))

$$\dot{x} = Ax + Bu$$

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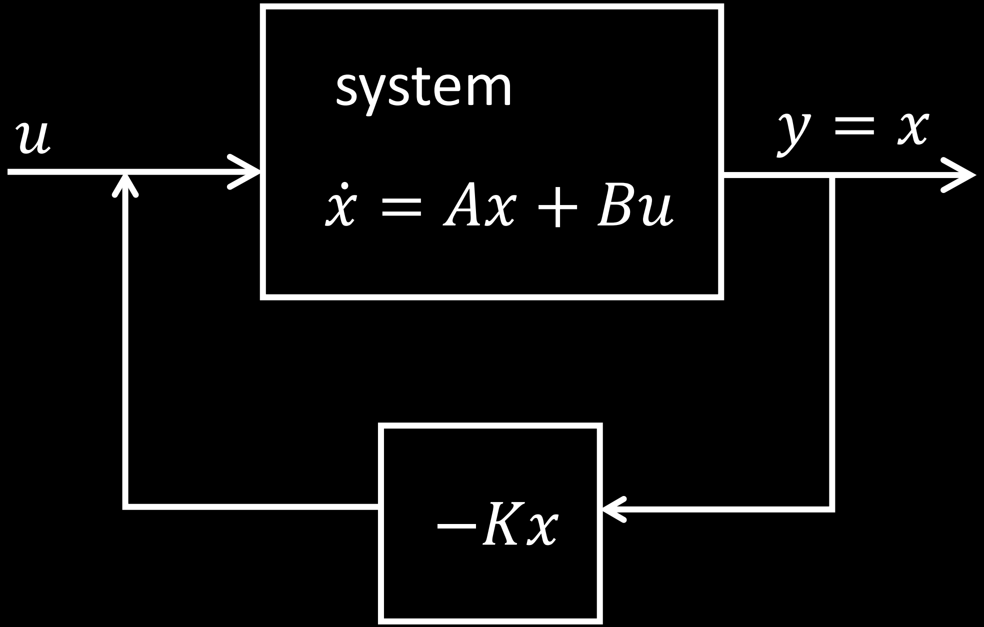

Controllability

- Can you control this system?

1. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ *uncontrollable*
   - There's no way to directly/indirectly affect x_1

- What could you change to make it controllable?
  - Add more control authority!

2. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *controllable*

$$\dot{x} = Ax + Bu$$

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- Can you control this system?

3. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ *controllable*

  - Systems with coupled dynamics can be controllable…
  - If A is tightly coupled, you can get away with a simple B and few sensors

$$\dot{x} = Ax + Bu$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

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  2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ *controllable*
  3. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *controllable*
- Matlab >> ctrb(A,B)
- Controllability matrix
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable

$$\dot{x} = Ax + Bu$$

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  2. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *controllable*
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  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable
- System 1:
  - $\mathbb{C} = [\qquad\qquad]$ ← *These still can't touch x_1!*
  - $\mathbb{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ rank=1, n=2

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$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

$x \epsilon \mathbb{R}^n$

$A \epsilon \mathbb{R}^{n \times m}$

$u \epsilon \mathbb{R}^q$

$B \epsilon \mathbb{R}^{n \times q}$

Controllability

- Can you control this system?
  1. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *uncontrollable*
  2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ *controllable*
  3. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *controllable*
- Matlab >> ctrb(A,B)
- Controllability matrix
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable
- System 1: $\mathbb{C} = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$ rank=1, n=2
- System 3:
  - $\mathbb{C} = \begin{bmatrix}0 & 1\cdot 0 + 1\cdot 1\\ 1 & 0\cdot 0 + 2\cdot 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}$ rank=2, n=2

$$\dot{x} = Ax + Bu$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

$x\epsilon\mathbb{R}^n$

$A\epsilon\mathbb{R}^{n\times m}$

$u\epsilon\mathbb{R}^q$

$B\epsilon\mathbb{R}^{n\times q}$

Fyi!

- Just because a linearized, nonlinear system is uncontrollable, it can still be nonlinearly controllable!

Controllability Matrix and the Discrete Time Impulse Response

$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

- Why does $\mathbb{C}$ predict controllability?!
- Discrete time impulse response: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$

(assume a single input actuator)

  - $u(0) = 1 \qquad x(0) = 0$
  - $u(1) = 0 \qquad x(1) = \tilde{A}x(0) + \tilde{B}u(0) = \tilde{B}$
  - $u(2) = 0 \qquad x(2) = \tilde{A}x(1) + \tilde{B}u(0) = \tilde{A}\tilde{B}$
  - $u(3) = 0 \qquad x(3) = \tilde{A}^2\tilde{B}$
  - $\dots \qquad \dots$
  - $u(m) = 0 \qquad x(m) = \tilde{A}^{m-1}\tilde{B}$

If the system is controllable, then the impulse response affects every state in $\mathbb{R}^n$

Prof. Kirstin Hagelskjær Petersen
kirstin@cornell.edu

Reachability

Controllabillity and Reachability

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

Reachability

- $\mathcal{R}_t$, states that are reachable at time t
- $\mathcal{R}_t = \{\xi \epsilon \mathbb{R}^n$ for which there is an input $u(t)$ that makes $\mathrm{x(t)} = \xi$

Equivalences

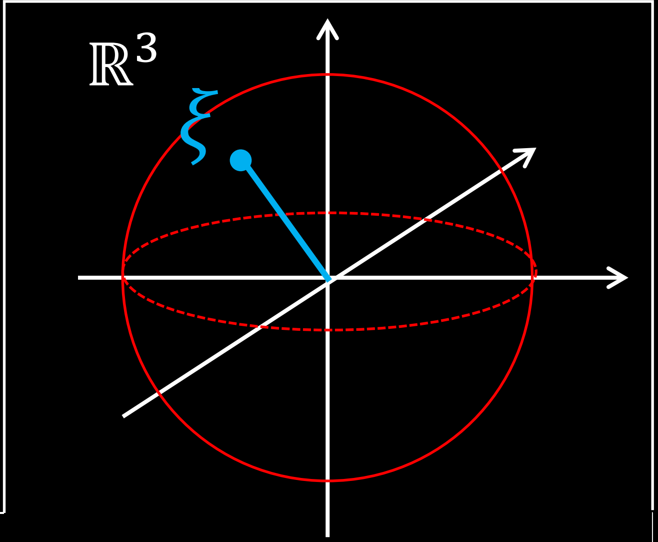
1. The system is controllable
   - iff $rank(\mathbb{C}) = n$
2. You can choose K to arbitrarily place the eigenvalues of your closed loop system
   - $\dot{x} = (A - BK)x$
3. You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
   - $\mathcal{R}_t = \mathbb{R}^n$

Controllabillity and Reachability

$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$

$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$

(if the point is reachable, any point in that direction is reachable)



Reachability

- $\mathcal{R}_t$, states that are reachable at time t
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Equivalences

1. The system is controllable
   - iff $rank(\mathbb{C}) = n$
2. You can choose K to arbitrarily place the eigenvalues of your closed loop system
   - $\dot{x} = (A - BK)x$ *
3. You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
   - $\mathcal{R}_t = \mathbb{R}^n$

>>K = scipy.signal.place_poles(*A*, *B*, *poles*)

Prof. Kirstin Hagelskjær Petersen
kirstin@cornell.edu

Controllability Gramians

- We can test if the system is controllable
- But not how easy it is to control
- …or in which directions it is easier
- How could you improve B?

Controllabillity Gramian

- $x(t) = e^{At}x(0) \;+ \int_0^t e^{A(t-\tau)}\,Bu(\tau)d\tau$

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

```
>> rank(ctrb(A,b))
>> [U,S,V] = svd(ℂ, ‘econ’)
```

- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \qquad W_t \epsilon \mathbb{R}^{nxn}$
  - Discrete time
    - $W_t \approx \mathbb{C}\mathbb{C}^T$
    - $W_t\xi = \lambda\xi$

The SVD of A takes the form: $A = U\Sigma V^T$
U = left singular vector
V = right singular vector
Σ = diagonal matrix with singular values

(The eigenvectors with the biggest eigenvalues of the controllability gramian, are also the most controllable directions in state space!)

Controllabillity Gramian

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

  (convolution of e^{At} with $u(\tau)$)

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

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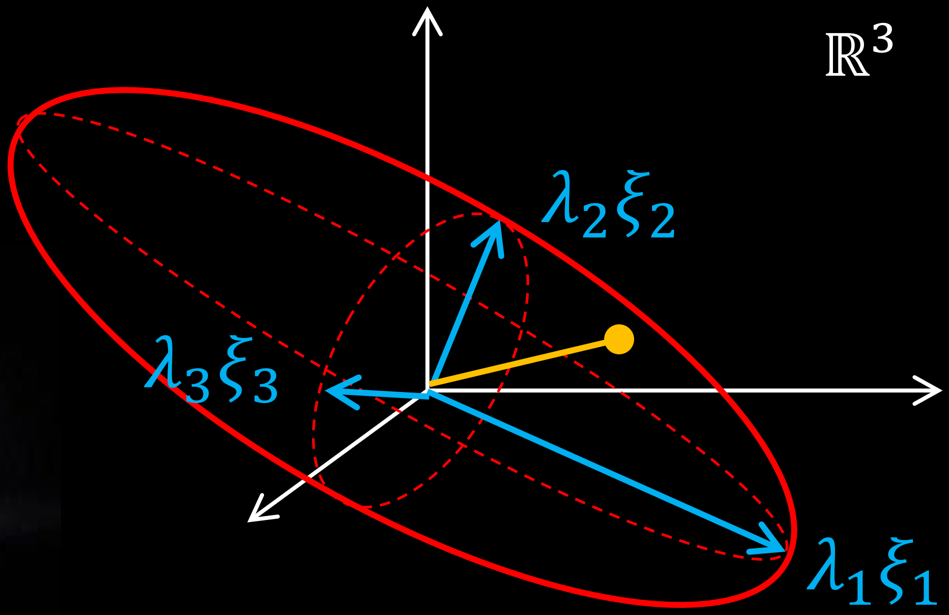

Controllabillity Gramian

By DLR, CC-BY 3.0, CC BY 3.0 de, https://commons.wikimedia.org/w/index.php?curid=61072555

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

```
>> rank(ctrb(A,b))
>> [U,S,V] = svd(ℂ, ‘econ’)
```

- Controllability for very high dimensional systems?
- Many directions in $\mathbb{R}^n$ are extremely stable - you only need to control directions that impact your control objective
- *Stabilizability*

Controllabillity Gramian

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

- $x(t) = e^{At}x(0) \; + \int_0^t e^{A(t-\tau)}\, Bu(\tau)d\tau$

  (convolution of e^{At} with $u(\tau)$)

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$$

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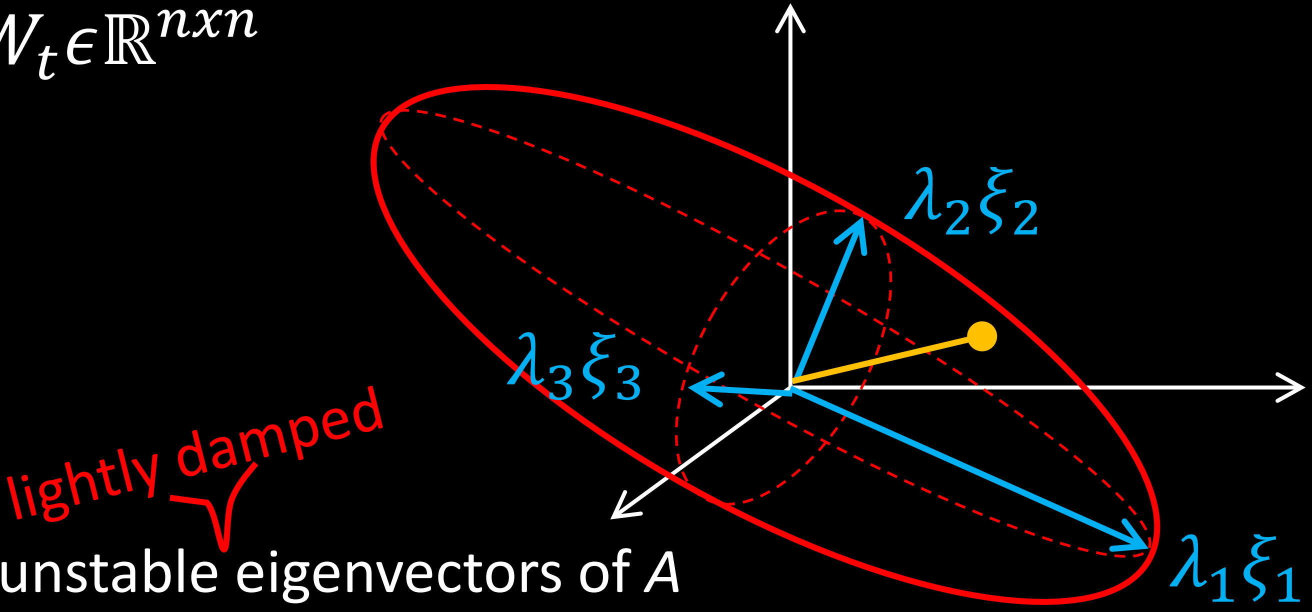
>> rank(ctrb(A,b))
>> [U,S,V] = svd(ℂ, 'econ')
```

- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \quad W_t \epsilon \mathbb{R}^{nxn}$
  - $W_t\xi = \lambda\xi$
  - $W_t \approx \mathbb{C}\mathbb{C}^T$
- *Stabilizability*
  - A system is stabilizable iff all unstable eigenvectors of *A* are in the controllable subspace

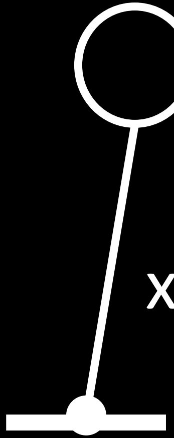


Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- LQR control
- Observability

$$\dot{x} = Ax + Bu$$



$$x=\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}$$

This should look familiar from..
- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

Engineering Communications

- Discussion of webpages and online portfolios
- Led by Traci M. Nathans-Kelly